REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 0704-0188*

1. REPORT DATE *(DD-MM-YYYY)*
11-05-2012

2. REPORT TYPE

3. DATES COVERED *(From - To)*

4. TITLE AND SUBTITLE
Enumerative Geometry of Hyperplane Arrangements

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Paul, Thomas Joseph

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U.S. Naval Academy
Annapolis, MD 21402

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)
Trident Scholar Report no. 408 (2012)

12. DISTRIBUTION/AVAILABILITY STATEMENT
This document has been approved for public release; its distribution is UNLIMITED

13. SUPPLEMENTARY NOTES

14. ABSTRACT
Systems of polynomial equations arise in a wide range of applications, from statistical economics to robot motion planning. Sometimes it's helpful just to count the number of solutions to the system of equations. In enumerative geometry we count the the number of geometric objects that satisfy a specific system of polynomial constraints. The goal of this project is to count the number of hyperplane arrangements sharing the same combinatorial type that also satisfy a list of geometric conditions. We develop explicit formulas for families of generic hyperplane arrangements in any dimension, as well as for families of pencils in the projective plane, in addition to calculating that there are 16,695 different braid arrangements of six lines through eight points in general position.

15. SUBJECT TERMS
Enumerative Geometry, Hyperplane Arrangements, Tutte Polynomial, Chow Ring, Braid Arrangement

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
				68	19b. TELEPHONE NUMBER *(Include area code)*

A TRIDENT SCHOLAR PROJECT REPORT

NO. 408

Enumerative Geometry of Hyperplane Arrangements

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USNA-1531-2

Abstract

Systems of polynomial equations arise in a wide range of applications, from statistical economics to robot motion planning. Sometimes it's helpful just to count the number of solutions to the system of equations. In enumerative geometry we count the the number of geometric objects that satisfy a specific system of polynomial constraints. The goal of this project is to count the number of hyperplane arrangements sharing the same combinatorial type that also satisfy a list of geometric conditions. We develop explicit formulas for families of generic hyperplane arrangements in any dimension, as well as for families of pencils in the projective plane.

Our main result is that there are 16695 different braid arrangements of 6 lines containing 8 fixed points in general position. Moreover, we determine the characteristic numbers of 3 generic lines. Using these numbers, we solve all possible intersection problems involving 3 generic lines. In order to compute these enumerative results, we use both combinatorial methods and intersection polynomials in the Chow ring. The enumerative results give deep information about the moduli space of arrangements with a fixed intersection type. Mnëv's Universality Theorem shows these spaces, the points of which are themselves individual arrangements, can be arbitrarily complicated (see Mnëv [18] and Vakil [30]); however, we use Macaulay2 to determine explicit minimal defining equations of the ideal of the Zariski closure of the moduli space of the braid arrangement.

We also develop code for the computer algebra package SAGE [27] to compute the multivariate Tutte polynomial After extensive computer trials in SAGE, we found relationships between evaluations of the Tutte polynomial and the solutions to enumerative problems for generic arrangements and pencils, but in general the precise connection between these is elusive. In another direction we count the number of combinatorial types of arrangements with six or fewer lines. These counts also enumerate the number of realizable rank-3 matroids.

Keywords

Enumerative Geometry

Hyperplane Arrangements

Tutte Polynomial

Chow Ring

Braid Arrangement

Contents

List of Notation

$\mathcal{A}$	A hyperplane arrangement
$\mathbb{C}^n$	n-dimensional complex space
Δ^k	Standard k-simplex
δ^k	k^{th} Coboundary map
$L(\mathcal{A})$	The intersection lattice of a hyperplane arrangment
$\mathcal{M}(L)$	Moduli space of L
$\mu^k\nu^{t-k}$	Characteristic number
$\mathbb{P}^n$	n-dimensional projective space
$\mathbb{R}^n$	n-dimensional real space
$\mathbb{V}$	Algebraic variety
$A_*(X)$	Chow Ring of X
$C_k(X)$	Group of singular k-chains
H	A hyperplane
$H^*(X)$	Cohomology ring of X
K	A field
$M(\mathcal{A})$	A matroid
V	A vector space
Z	Integers

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1 Introduction

Enumerative geometry is the field of mathematics concerned with counting the number of geometric objects that satisfy some specified conditions. Enumerative problems have been a driving force throughout the history of mathematics, dating back to the Problem of Apollonius (see Smith et al. [24]) and Pappus' Theorem (see Richter-Gebert [21]). The quest to answer these problems has propelled the areas of algebraic geometry, commutative algebra, combinatorics, and algebraic topology, to name a few. Hilbert's 15^{th} problem asks for a rigorous treatment of Schubert's calculus, which is still an active area of research with exciting connections to string theory (see Hilbert [13], Kleiman [17], and Katz [16]). Enumerative problems are often quite easy to state, but in many cases prove very difficult to solve.

In this paper, we answer enumerative questions concerning hyperplane arrangements. A hyperplane H is a vector space V is a codimension-1 linear subspace of V, and a hyperplane arrangement $\mathcal{A}$ is a finite collection of hyperplanes (see Orlik and Terao for general reference on hyperplane arrangements [20]). Each hyperplane arrangement $\mathcal{A}$ has an intersection lattice $L(\mathcal{A})$, which is the set of all intersections of the hyperplanes in $\mathcal{A}$, ordered by reverse inclusion. Assuming our hyperplanes contain the origin in V, we can view an arrangement projectively in $\mathbb{P}^n$, which we discuss in Sections 2.1 and 4. While enumerative problems in $\mathbb{P}^1$ are trivial, there are extremely difficult and open problems in the projective plane $\mathbb{P}^2$, in which hyperplanes are lines (see Rimányi et al. [22]). The main question of this project is: how many distinct line arrangements with a fixed intersection lattice L contain d points in general position in $\mathbb{P}^2$? We develop explicit formulas to answer this question for the celebrated intersection lattices of generic arrangements, pencils, and the braid arrangement. The techniques used range from combinatorial counting methods (see Dekking et al. [3]) to intersection polynomials in the Chow ring (see Fulton [7]) to evaluations of the multivariate Tutte polynomial (see Sokal [25]). The intersection polynomial in the Chow ring has been used to solve enumerative problems before, but not for problems involving hyperplane arrangements. Similarly, as far as we know this is the first use of the multivariate Tutte polynomial to compute enumerative results for hyperplane arrangements.

Our most difficult result, Theorem 6.1.1, counts the number of braid arrangements through 8 points. The braid arrangement has a number of very important real-world applications. The complement of the braid arrangement can be used for robot motion planning to prevent collisions (see Ghrist [8]). The complement of the braid arrangement is also useful in proving Arrow's Impossibility Theorem, which can be tersely summarized as saying that a dictatorship is the only fair voting system (see Arrow [1] and Terao [29]). The main result, which we verify using two independent methods, is that the number of distinct braid arrangements through 8 points in general position is 16695, as shown in Theorem 6.1.1. Constructing the intersection polynomial for the braid arrangement and interpreting its results requires great care, because there is "excess" intersection. We also determine the minimal defining equations for the moduli space of the braid arrangement, which is the set of all arrangements with that lattice type, in Section 6.3. This process involves taking the Zariski closure of the moduli space and finding the corresponding ideal. Rimányi et al. [22]

considered an equivalent problem, and determined a collection of defining equations. In Section 6.3 we show that some of these defining equations are redundant, and use to Macaulay2 [9] to provide a list of non-redundant equations.

In addition to answering enumerative problems of line arrangements passing through points in general position, we also generalize these problems to include tangency conditions imposed by curves of arbitrary degree. To do this, we compute the characteristic numbers for particular intersection lattices and then substitute these values into the tangency polynomial described in Section 7 (the results of this section are inspired by the treatment of enumerative problems in [7, Section 10.4]).

This project makes extensive use of the computer algebra packages SAGE and Macaulay2 for computations, as well as the program Geogebra to visualize complicated arrangements. In Appendix A we provide the code for programs we wrote to aid in solving enumerative problems, including code that generates the multivariate Tutte polynomial for any intersection lattice based on the independent sets in its matroid. Once this polynomial is created, another program tests for potential solutions to the enumerative question. To determine the independent sets in a matroid, we expanded upon SAGE code developed by Professor David Joyner, which in its original form was hard coded to generate the independent sets of arrangements of three lines, to include arrangements of an arbitrary number of lines.

2 Basic Definitions

We begin our discussion of this project by introducing key terms, concepts, and notation that will be used throughout.

A *commutative ring* is a set of elements closed under two operations, which are normally called addition and multiplication, that forms an Abelian group with respect to addition (i.e. a set of elements with an operation that has a unit, has inverses, and is associative and commutative) and is associative, commutative, and distributive under multiplication. A *field* is a commutative ring where every element has a multiplicative inverse. We mainly work with the fields $\mathbb{R}$ of real numbers and $\mathbb{C}$ of complex numbers.

2.1 Hyperplane Arrangements

A *vector space* V over a field K is a group V that admits a distributive action of the field K, which determines an element $cv \in V$ for each $c \in K$ and $v \in V$. While many of our results can be obtained over any coefficient field, they are stronger over the complex numbers, so we fix

$K = \mathbb{C}$. We restrict our attention to finite-dimensional vector spaces and we identify V with $\mathbb{C}^n$ by choosing a basis for V; however, for examples defined over the real numbers, we do not lose much when visualizing them in $\mathbb{R}^n$ compared to $\mathbb{C}^n$. A *hyperplane* H in an n-dimensional vector space V is the solution set of a linear equation. Hyperplanes are translations of codimension-1 linear subspaces of V. A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of hyperplanes.

Example 2.1.1. Fix the dimension $n = 3$ and let $H_1 = \{(x, y, z) : x = 0\}$ (abbreviated $H_1 = \{x = 0\}$), $H_2 = \{y = 0\}$, $H_3 = \{x - y = 0\}$, and $H_4 = \{x + y - z = 0\}$. Then let $\mathcal{A} = \{H_1, H_2, H_3, H_4\}$. Figure 1 depicts the real picture of this arrangement.

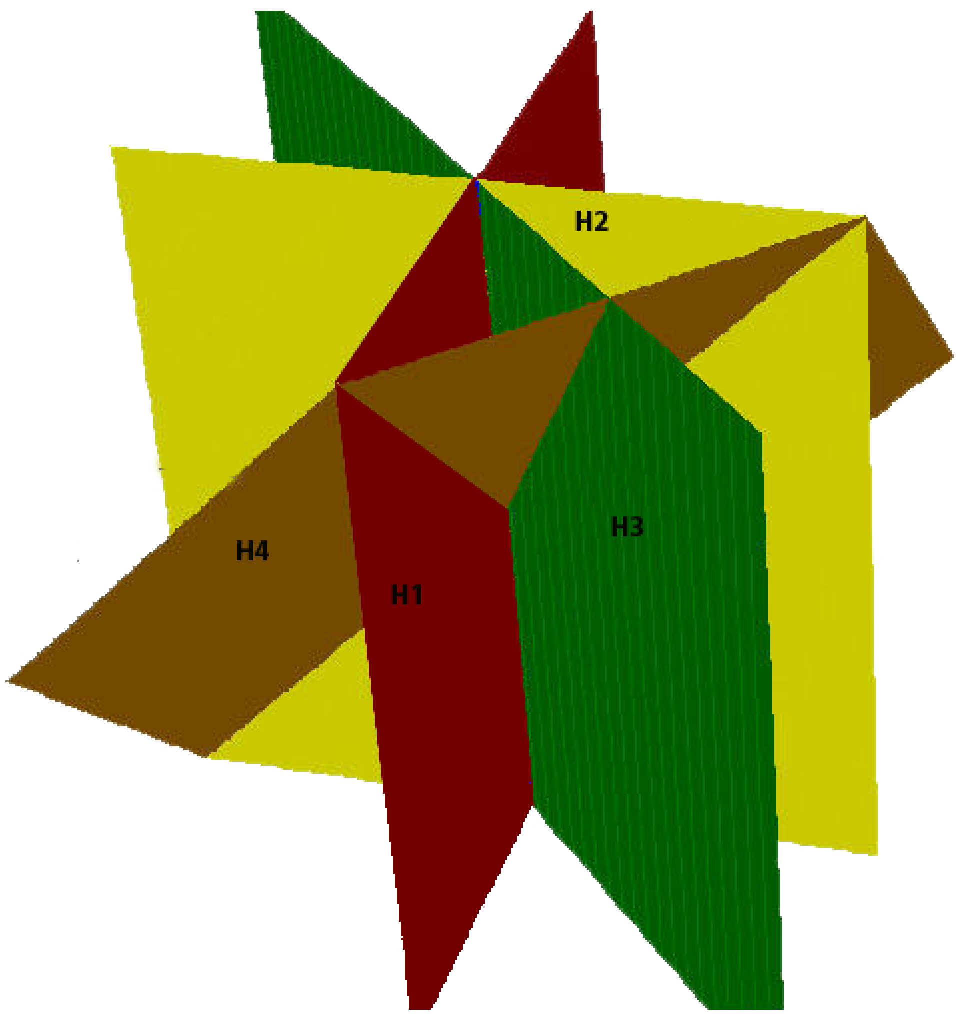

Figure 1: Hyperplane arrangement of four planes in $\mathbb{R}^3$

Working in an arbitrary vector space can be difficult because hyperplanes can be parallel and therefore not intersect. To avoid this complication, we work in projective space.

Definition 2.1.2. Fix a field K and an n-dimensional vector space V over K. The projective space $\mathbb{P}_K^{n-1}$ of V over K is the set of all lines through the origin in V. To emphasize the field when $K = \mathbb{R}$ or $\mathbb{C}$, we sometimes write $\mathbb{RP}^{n-1}$ or $\mathbb{CP}^{n-1}$, respectively. In the rest of the paper, we usually write $\mathbb{P}^{n-1}$ for $\mathbb{CP}^{n-1}$ since this is the projective space in which we are most interested.

In this project we often work in 2-dimensional complex projective space $\mathbb{CP}^2$. Projective space is so useful because any pair of lines in projective space intersect, whereas in $\mathbb{R}^2$ or $\mathbb{C}^2$ there exist pairs that do not intersect. So even if two lines in $\mathbb{P}^2$ appear to be parallel locally, they in fact intersect at infinity, much as two parallel railroad tracks appear to meet at the horizon. There is a relatively simple way to think of projectivizing a space like $\mathbb{R}^n$ or $\mathbb{C}^n$. As an example, projectivize $\mathbb{R}^3$ to get $\mathbb{RP}^2$ as follows. We know that $\mathbb{RP}^2$ consists of lines in $\mathbb{R}^3$ through the origin. However, there is a large piece of $\mathbb{RP}^2$ that consists of lines not contained in the plane $z = 0$. All such lines meet the plane $z = 1$ and we can identify the line – the point in $\mathbb{RP}^2$ – by the x and y coordinates of this point of intersection. So there is a big piece of $\mathbb{RP}^2$ that looks like $\mathbb{R}^2$. The remaining points of $\mathbb{RP}^2$ are lines in the plane $z = 0$, that is, points in a copy of $\mathbb{RP}^1$. In this way, $\mathbb{RP}^2 = \mathbb{R}^2 \cup \mathbb{RP}^1$. Here the points in $\mathbb{RP}^1$ are viewed as points at infinity.

Each line through the origin in $\mathbb{R}^3$ is determined by a direction vector, a nonzero vector that is parallel to the line. However, many vectors correspond to the same line, and all of these vectors are nonzero scalar multiples of one another. This gives rise to homogeneous coordinates on $\mathbb{RP}^2$. The line parallel to vector $\langle a, b, c\rangle$ has coordinates $[a : b : c]$. Of course, $[2a : 2b : 2c] = [a : b : c]$ because the parallel vectors $\langle 2a, 2b, 2c\rangle$ and $\langle a, b, c\rangle$ determine the same line (for more details, see Smith [24]).

Example 2.1.3. The plane $x + y + z = 0$ intersects $z = 1$ in the line $L_1 : x + y + 1 = 0$, that is the line $y = -x - 1$. This is why we say that $x + y + z = 0$ is the homogenization of L_1. Also, the plane $x + y + 2z = 0$ intersections $z = 1$ in the line $y = -x - 2$, a parallel line L_2. The two lines l_1 and L_2 do not meet in $\mathbb{R}^2$, but their homogenizations meet at $[1 : -1 : 0] \in \mathbb{RP}^2$.

If points in $\mathbb{P}^2$ are lines in $\mathbb{R}^3$, it stands to reason that lines in $\mathbb{P}^2$ are planes in $\mathbb{R}^3$. Figure 2 depicts the projectivization of the hyperplane arrangement in Figure 1, with its lines and points of intersection labeled.

Projective space can also be thought of as the disjoint union of a number of smaller spaces. Start with $\mathbb{RP}^2$, or real projective 2-space. This can be constructed by taking a copy of $\mathbb{R}^2$, wrapping an $\mathbb{R}^1$ around it at infinity, and then closing the $\mathbb{R}^1$ with an $\mathbb{R}^0$ (to be precise, the $\mathbb{R}^1$ wraps twice around $\mathbb{R}^2$). Written more explicitly, $\mathbb{RP}^2 = \{[x : y : z] \in \mathbb{RP}^2 : z \neq 0\} (\cong \mathbb{R}^2) \cup \{[x : y : 0] \in \mathbb{RP}^2 : y \neq 0\} (\cong \mathbb{R}^1) \cup \{[x : 0 : 0] \in \mathbb{RP}^2 : x \neq 0\} (\cong \mathbb{R}^0)$. More generally,

$$\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C}^1 \cup \mathbb{C}^0.$$

Another interesting property of projective space is its *dual*. Let $p = [a : b : c]$ be a point in $\mathbb{P}^2$. Then there exists a vector $v = \langle a, b, c\rangle$ in $\mathbb{R}^3$ representing this point. Recall that a plane can be

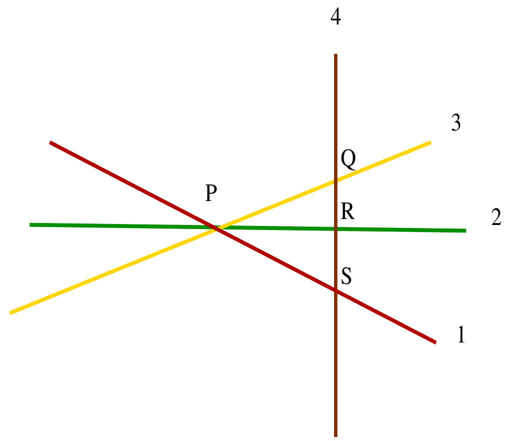

Figure 2: Hyperplane arrangement of four lines in $\mathbb{P}^2$

determined from a point and a normal vector. For this plane to determine an object in projective space, it must pass through the origin in $\mathbb{R}^3$, so now we have a point $(0, 0, 0)$ on the hyperplane and a normal vector $\langle a, b, c \rangle$ which fix a specific plane in $\mathbb{R}^3$. The image in $\mathbb{P}^2$ of this hyperplane of $\mathbb{R}^3$ is a line $\hat{p}$ in $\mathbb{P}^2$, which we call the dual of p. The equation of this line is $ax + by + cz = 0$. Note that scaling the coefficients of this line by a nonzero multiple does not change the line, so each line in $\mathbb{P}^2$ can itself be viewed as a point in some copy of $\mathbb{P}^2$, which we call the dual of $\mathbb{P}^2$. We denote the dual of $\mathbb{P}^2$ by $\mathbb{P}^{2*}$. We emphasize that the dual of a point in $\mathbb{P}^2$ is a line in $\mathbb{P}^2$ (that is, a point in $\mathbb{P}^{2*}$). Similarly, the dual of a point in $\mathbb{P}^{2*}$ is a line in $\mathbb{P}^{2*}$, which can be interpreted as a point in $\mathbb{P}^2$. In this way, if we dualize a point and then dualize the result, we obtain the original.

Example 2.1.4. The point $p = [2 : 1 : 3]$ dualizes to the line $2x + y + 3z = 0$. This is the point $[2 : 1 : 3]$ in $\mathbb{P}^{2*}$. Let $[a : b : c]$ be the homogeneous coordinates in $\mathbb{P}^{2*}$ so that $ax + by + cz = 0$ is the line in $\mathbb{P}^2$ corresponding to $[a : b : c] \in \mathbb{P}^{2*}$. A line L in $\mathbb{P}^{2*}$ looks like

$$Xa + Yb + Zc = 0$$

for complex numbers X, Y, and Z not all zero. This line dualizes to the point $[X : Y : Z] \in \mathbb{P}^{2**} \cong \mathbb{P}^2$. Note that every point on the line L corresponds to a line in $\mathbb{P}^2$ through the point $[X : Y : Z] \in \mathbb{P}^2$. Thus, requiring a line to pass through a fixed point in $\mathbb{P}^2$ restricts our attention to lines corresponding to points on a hyperplane in $\mathbb{P}^{2*}$.

Also critical in this project is the *intersection lattice* of a hyperplane arrangement $\mathcal{A}$.

Definition 2.1.5. Given an arrangement $\mathcal{A}$, let $L(\mathcal{A})$ be the set of the intersections of the elements of $\mathcal{A}$. We define a partial order on $L(\mathcal{A})$ by reverse inclusion: for all $X, Y \in L(\mathcal{A})$, $X \preceq Y \iff X \supseteq Y$. The *intersection lattice* of $\mathcal{A}$ is the poset $(L(\mathcal{A}), \preceq)$. When no confusion can result, we just write $L(\mathcal{A})$ for the intersection lattice of $\mathcal{A}$.

Remark 2.1.6. If X and Y are two elements of $L(\mathcal{A})$ then their least upper bound $X \vee Y$ is the smallest element of $L(\mathcal{A})$ such that $X \preceq X \vee Y$ and $Y \preceq X \vee Y$. Similarly, their greatest lower bound $X \wedge Y$ is the largest element of $L(\mathcal{A})$ such that $X \wedge Y \preceq X$ and $X \wedge Y \preceq Y$. The poset $L(\mathcal{A})$ is called the intersection *lattice* because least upper bounds and greatest lower bounds between any two elements always exist: $X \vee Y = X \cap Y$ and $X \wedge Y = \cap\{Z \in L : \ X \cup Y \subseteq Z\}$. Moreover, for any arrangement $\mathcal{A}$, $L(\mathcal{A})$ is actually a ranked atomic geometric lattice (see Orlik and Terao [20]), but we don't need this extra structure for this project.

A hyperplane arrangement's intersection lattice encodes geometric properties of the arrangement. The objects on each level of the intersection lattice have the same codimension, and the lines connecting the elements of the intersection lattice indicate containment. Figure 3 is the intersection lattice for the hyperplane arrangement in Figure 2.

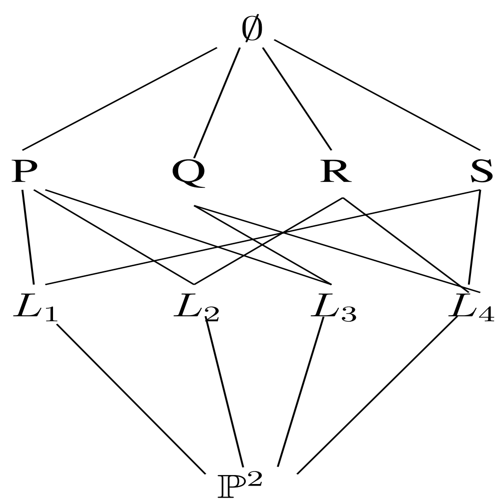

Figure 3: Intersection lattice for Figure 2

Remark 2.1.7. As the number of lines ℓ increases, the number of possible different intersection lattice types increases very quickly. We were able to generate pictures of all possible intersection lattice types for $\ell \leq 7$, but for $\ell > 7$ this is prohibitively time-consuming. The number of intersection lattice types for $\ell \leq 12$ are displayed in Table 1.

Number of Lines ℓ	Intersection Types
1	1
2	1
3	2
4	3
5	5
6	10
7	24
8	69
9	384
10	5250
11	232929
12	28872973

Table 1: Number of Realizable Intersection Lattice Types for ℓ lines

Figure 4 shows the 10 different intersection lattices for arrangements of six lines.

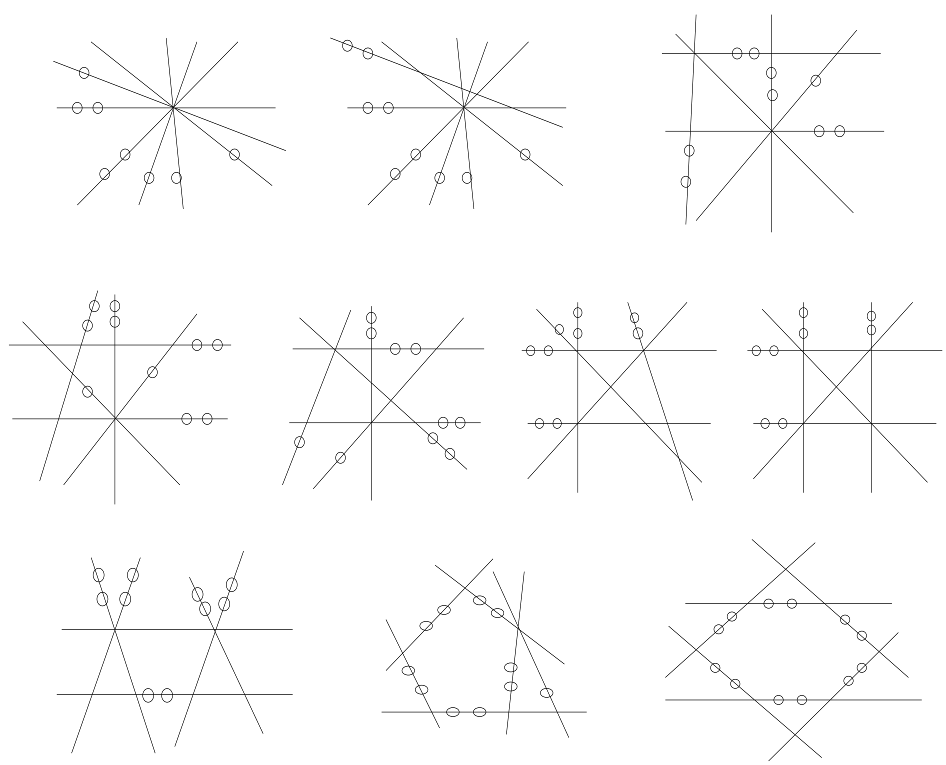

Figure 4: 10 Intersection Lattice Types for $\ell = 6$ lines

A *matroid* is another way to encode combinatorial information about a hyperplane arrangement (see Stanley [26]).

Definition 2.1.8. A matroid M is a pair (E,I), where E is a finite set called the ground set and I is a collection of subsets of E called the independent sets such that:

1. The empty set is independent, i.e. $\emptyset \in I$.

2. Every subset of an independent set is independent, i.e. for each $A' \subseteq A \subseteq E, A \in I \Rightarrow A' \in I$.

3. If A and B are two independent set of I and A has more elements than B, then there exists an element in A which is not in B that when added to B still gives an independent set.

Using the hyperplane arrangement in Figure 2, we define a matroid where the ground set E is $\{L_1, L_2, L_3, L_4\}$ and the set of independent sets I is

$$\{\{L_1\}, \{L_2\}, \{L_3\}, \{L_4\}, \{L_1L_2\}, \{L_1L_3\}, \{L_1L_4\}, \{L_2L_3\}, \{L_2L_4\}, \{L_3L_4\},$$

$$\{L_1L_2L_4\}, \{L_1L_3L_4\}, \{L_2L_3L_4\}, \{L_1L_2L_3L_4\}\}.$$

In general, given an arrangement $\mathcal{A}$ of ℓ labeled hyperplanes, define the matroid $M(\mathcal{A})$ to be a matroid whose ground set E is the set of hyperplanes in $\mathcal{A}$. As well, define a subset $B \subseteq E$, with k elements, to be an independent set of $M(\mathcal{A})$ if the hyperplanes in B intersect in a linear space of codimension k.

It follows that the matroid $M(\mathcal{A})$ for the previous example is $M(\mathcal{A}) =$ $(\{L_1, L_2, L_3, L_4\}, \mathcal{P}(\{L_1, L_2, L_3, L_4\}) \setminus \{L_1, L_2, L_3\})$, where $\mathcal{P}(S)$ denotes the powerset of a set S, the set of all possible subsets of S.

2.2 Topology, Homology, and Cohomology

Now that we've defined some basic terms regarding hyperplane arrangements, we explore a few important concepts in geometry.

We begin our discussion of enumerative geometry by defining a fundamental mathematical concept, a *topological space*. A topological space is a set X together with a collection T of subsets of X, satisfying the following axioms:

1. The empty set and X are in T.
2. T is closed under arbitrary union.
3. T is closed under finite intersection.

In this case, T is called a topology on X and the elements of T are called open sets of X.

Example 2.2.1. The Euclidean topology on the set of real numbers $\mathbb{R}$ is generated by the basis consisting of open intervals (x, y) such that $x < y$. That is to say, any open set in the Euclidean topology is the union or finite intersection of open intervals.

In our enumerative study of hyperplane arrangements, we encounter some very complicated higher-dimensional topological spaces. To find solutions to these enumerative problems it is helpful to understand the overall structure, or global geometry, of these spaces. Before we define these spaces, we present a useful tool in describing the global geometry of an arbitrary topological space: *homology* and *cohomology*.

There are a number of different homology and cohomology theories associated with various types of spaces. Usually these theories involve a sequence of groups with maps between them, called a chain complex, with the homology groups taking the form of quotient groups $\frac{cycles}{boundaries} = \frac{kernel}{image}$. For our enumerative questions, we often deal with the singular homology of a topological space. To define singular homology, we must develop some notation. The first term we define is the *k-simplex*.

Definition 2.2.2. The standard k-simplex Δ^k is the subset of $\mathbb{R}^{k+1}$ given by

$$\Delta^k = \left\{ (t_0, \ldots, t_k) \in \mathbb{R}^{k+1} | \sum_{i=0}^{k} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}.$$

A k-simplex can be thought of as the generalization of a triangle or tetrahedron to k dimensions.

Example 2.2.3. Let $k = 0$. The 0-simplex lies in $\mathbb{R}^1$ (the number line), and is the point $\Delta^0 = \{(t_0) \in \mathbb{R}^1 | \sum_{i=0}^{0} t_i = 1\} = \{(1)\}$. A 0-simplex is displayed in Figure 5.

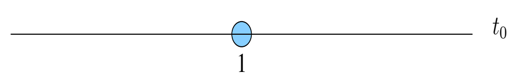

Figure 5: 0-simplex

Let $k = 1$. The 1-simplex is $\Delta^1 = \{(t_0, t_1) \in \mathbb{R}^2 | \sum_{i=0}^{1} t_i = 1, t_0 \geq 0, t_1 \geq 0\}$, which is the line segment in $\mathbb{R}^2$ from $(1, 0)$ to $(0, 1)$. A 1-simplex is displayed in Figure 6.

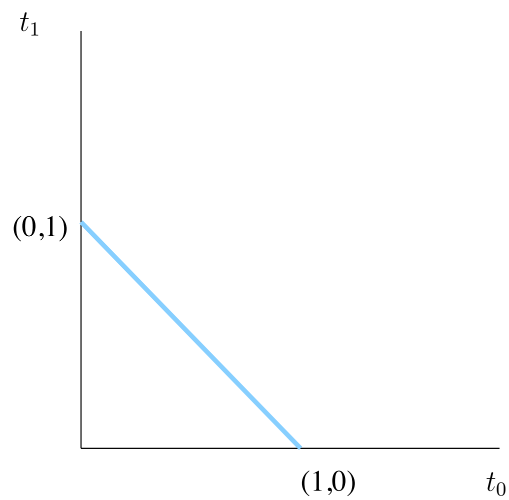

Figure 6: 1-simplex

Let $k = 2$. The 2-simplex, $\Delta^2 = \{(t_0, t_1, t_2) \in \mathbb{R}^3 | \sum_{i=0}^{2} t_i = 1; t_0, t_1, t_2 \geq 0\}$, is the triangle with vertices at $(1, 0, 0)$,$(0, 1, 0)$, and $(0, 0, 1)$. A 2-simplex is displayed in Figure 7.

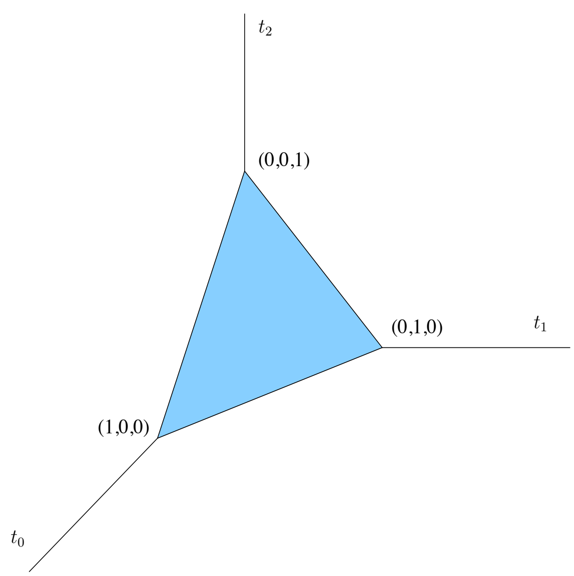

Figure 7: 2-simplex

The pattern here is fairly obvious, with the dimension and number of vertices increasing by 1 each time k increases, but the pictures become difficult to draw.

It is also important to understand the concept of a *continuous function* as it applies to topology. A continuous function is one such that the inverse image of an open set is open. A continuous function $f : X \to Y$ between topological spaces is a homeomorphism if f is a bijection and $f^{-1} : Y \to X$ is continuous. The topology on a k-simplex is induced from the topology of the ambient $\mathbb{R}^{k+1}$: open sets on Δ^k are all of the form $U \cap \Delta^k$ where U is an open set in $\mathbb{R}^{k+1}$. With this topology, the simplex is homeomorphic to a k-dimensional ball.

Now it is possible for us to define *singular k-chains* and *k-cochains*.

Definition 2.2.4. Let X be a topological space. The group of singular k-chains $C_k(X)$ is the free Abelian group generated by the set of all continuous maps from the standard k-simplex to X,

$$C_k(X) = \left\{ \sum_i n_i f_i | n_i \in \mathbb{Z}, f_i : \Delta^k \to X \text{ continuous} \right\}.$$

The group of k-cochains is $C^k(X) =$Hom$(C_k(X), \mathbb{Z})$, the homomorphisms from $C_k(X)$ to $\mathbb{Z}$.

Let the *boundary map* be $\delta_k : C_k(X) \to C_{k-1}(X)$,

$$\sum_r n_r f_r \mapsto \sum_{r,i} (-1)^i n_r (f_r \circ j_i),$$

such that $j_i : \Delta^{k-1} \hookrightarrow \Delta^k$ is the map that sends a standard $(k-1)$-simplex to the i^{th} face of a standard k-simplex

$$(v_0, \ldots, v_k) \mapsto (v_0, \ldots, 0, \ldots, v_k),$$

where the 0 is in the $(i+1)^{\text{th}}$ entry.

Let the *coboundary map* $\delta^k : C^k(X) \to C^{k+1}(X)$ be defined by

$$(\delta^k(f))(Z) = f(\delta_{k+1}(Z)).$$

The group $Ker(\delta^k)$ is called the group of *k-cocycles* $Z^k(X)$ and the group $Im(\delta^{k-1})$ is called the group $B^k(X)$ of *k-coboundaries*.

Now that we have explored some of its technical components, we can define the k^{th}*cohomology group* of X.

Definition 2.2.5. The k^{th} cohomology group of a topological space X is the quotient

$$H^k(X) = \frac{Z^k(X)}{B^k(X)}.$$

Example 2.2.6. Let X be a point p. The k^{th} cohomology group of p is

$$H^k(p,\mathbb{Z}) = \left\{ \begin{array}{cc} \mathbb{Z} & \text{if } k=0 \\ 0 & \text{if } k>0 \end{array} \right\}.$$

To see this, we begin by writing down the group of singular k-chains:

$$\begin{aligned} C_k(p) &= \left\{ \sum_i n_i f_i | n_i \in \mathbb{Z}, f_i : \Delta^k \to p \right\} \\ &= \left\{ n f_k | n \in \mathbb{Z}, f_k : \Delta^k \to p \text{ continuous} \right\} \\ &\cong \mathbb{Z} f_k \end{aligned}$$

There is only one map from a k-simplex Δ^k to a point p, which we call f_k, and because we can choose to multiply f_k by any integer n, $C_k(p)$ is isomorphic to $\mathbb{Z}f_k$. Now we can find the group of k-cochains:

$$\begin{aligned} C^k(p) &= \text{Hom}(C_k(p),\mathbb{Z}) \\ &= \text{Hom}(\mathbb{Z}f_k,\mathbb{Z}). \end{aligned}$$

The map $\phi \in \text{Hom}(\mathbb{Z}f_k,\mathbb{Z})$ is completely determined by $\phi(f_k) \in \mathbb{Z}$, so there is a group isomorphism such that $C^k(p) \cong \mathbb{Z}$ for all k.

Next we construct our coboundary maps. These k-coboundary maps take homomorphisms of $\phi_k : \Delta^k \to p$ to homomorphisms $\delta^k(\phi_k) : \Delta^{k+1} \to p$, as shown in (1).

$$\begin{array}{ccccccccc} 0 & \xrightarrow{\delta^{-1}} & C^0(p) & \xrightarrow{\delta^0} & C^1(p) & \xrightarrow{\delta^1} & C^2(p) & \xrightarrow{\delta^2} & \cdots \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \end{array} \tag{1}$$

Let $\phi_0 \in C^0(p)$. Then $\delta^0(\phi_0) \in C^1(p) = \text{Hom}(\mathbb{Z}f_1,\mathbb{Z})$, so $\delta^0(\phi_0)$ is completely determined by the value of $(\delta^0(\phi_0))(f_1)$. Now

$$(\delta^0(\phi_0))(f_1) = \phi_0(\delta_1(f_1)).$$

But $\delta_1(f_1) = f_1 \circ j_0 - f_1 \circ j_1$, where

$$\begin{array}{rrcccccc} & f_1 \circ j_0 & : & \Delta^0 & \xrightarrow{j_0} & \Delta^1|_0 & \xrightarrow{f_1} & p \\ \text{and} & f_1 \circ j_1 & : & \Delta^0 & \xrightarrow{j_1} & \Delta^1|_1 & \xrightarrow{f_1} & p. \end{array} \tag{2}$$

From (2) we see that $f_1 \circ j_0 : \Delta^0 \to p$ and $f_1 \circ j_1 : \Delta^0 \to p$ are both f_0, so $f_1 \circ j_0 = f_1 \circ j_1 = f_0$. Therefore, δ^0 sends f_1 to $f_1 \circ j_0 - f_1 \circ j_1 = 0(f_0)$, so $(\delta^0(\phi_0))(f_1) = 0(f_0)$ and $\delta^0(\phi_0) = 0$ for all $\phi_0 \in C^0(p)$. That is, $\delta^0 : C^0(p) \to C^1(p)$ is the zero homomorphism. Then $\text{Ker}(\delta^0) = C^0(p) = \mathbb{Z}f_0$ because δ^0 is the zero map, and $\text{Im}(\delta^{-1}) = 0$ because the only element of its domain is 0. From this information,

$$H^0(p, \mathbb{Z}) = \frac{\text{Ker}(\delta^0)}{\text{Im}(\delta^{-1})} \cong \frac{\mathbb{Z}f_0}{0} \cong \mathbb{Z}.$$

To compute the next coboundary map, let $\phi_1 : \mathbb{Z}f_1 \to \mathbb{Z}$ be an element of $C^1(p)$. Then, $\delta^1(\phi_1) : \mathbb{Z}f_2 \to \mathbb{Z}$, is constructed in much the same way as δ^0. We know that $\delta^1(\phi_1)$ is a map sending $\mathbb{Z}f_2$ to $\mathbb{Z}$ that sends f_2 to $\sum_{i=0}^{2}(-1)^i f_2 \circ j_i = f_2 \circ j_0 - f_2 \circ j_1 + f_2 \circ j_2$. Each of the $f_2 \circ j_i$ is a composition $\Delta^1 \to \Delta^2|_i \to p$, therefore they are all the map from Δ^1 to the point p, that is the map f_1. Since each of these maps is the same, $f_2 \circ j_0 - f_2 \circ j_1 + f_2 \circ j_2 = f_1$. This means that δ^1 is the map sending f_2 to f_1. Identifying $\mathbb{Z}f_2$ and $\mathbb{Z}f_1$ with $\mathbb{Z}$, we see that δ^1 is identified with the map

$$\begin{array}{ccccccc} \mathbb{Z} & \cong & \mathbb{Z}f_2 & \xrightarrow{\delta^1} & \mathbb{Z}f_1 & \cong & \mathbb{Z} \\ 1 & \longmapsto & 1f_2 & \longmapsto & 1f_1 & \longmapsto & 1, \end{array} \tag{3}$$

so δ^1 is the identity map.

$\text{Ker}(\delta^1) = 0$ because the only thing the identity map takes to zero is zero, and $\text{Im}(\delta^0) = 0$ because δ^0 is the zero map. Therefore,

$$H^1(p, \mathbb{Z}) = \frac{\text{Ker}(\delta^1)}{\text{Im}(\delta^0)} \cong \frac{0}{0} \cong 0.$$

Using this method to find all of the coboundary maps, we see a pattern arise. When k is odd, δ^k is the identity map, and when k is even, δ^k is the zero map. From this, we can say that for odd k, $H^k(p, \mathbb{Z}) = \frac{\text{Ker(Identity)}}{\text{Im(Zero)}} = \frac{0}{0} = 0$, and for even $k > 0$, $H^k(p, \mathbb{Z}) = \frac{\text{Ker(Zero)}}{\text{Im(Identity)}} = \frac{\mathbb{Z}}{\mathbb{Z}} = 0$. Therefore, for all $k > 0$, $H^k(p, \mathbb{Z}) = 0$.

Remark 2.2.7. It turns out that if two topological spaces are homotopy equivalent, then their homology and cohomology groups are isomorphic. For details and a proof, see Corollary 17 of Basener [2]. Since $\mathbb{R}^n$ is homotopy equivalent to a point,

$$H^k(\mathbb{R}^n, \mathbb{Z}) = \left\{ \begin{array}{ll} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{array} \right\}.$$

We won't make additional use of homotopy equivalence, so we don't give any further details.

Let X be a topological space. Then there is a multiplication operation $H^k(X, \mathbb{Z}) \times H^\ell(X, \mathbb{Z}) \to H^{k+\ell}(X, \mathbb{Z})$ that is compatible with the group operation on $\oplus H^k(X, \mathbb{Z})$ in such a way that the cohomology ring $H^*(X, \mathbb{Z}) = \oplus H^k(X, \mathbb{Z})$ becomes a graded ring. The multiplication operation is called the cup product. To define the cup product on cohomology, we first define it on cochains.

Definition 2.2.8. Given two singular cochains $\phi \in C^k(X, \mathbb{Z})$ and $\psi \in C^\ell(X, \mathbb{Z})$, the cup product $\phi \cup \psi \in C^{k+\ell}(X, \mathbb{Z})$ is the cochain whose value on a singular simplex $\sigma : \Delta^{k+\ell} \to X$ is given by the formula

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_k,\ldots,v_{k+\ell}]})$$

where the right-hand side is the product of two integers.

Lemma 2.2.9. $\delta^{k+\ell}(\phi \cup \psi) = \delta^k(\phi) \cup \psi + (-1)^k \phi \cup \delta^\ell(\psi)$ *for* $\phi \in C^k(X, \mathbb{Z})$ *and* $\psi \in C^\ell(X, \mathbb{Z})$.

Proof. See Hatcher [12, lemma 3.6 on page 206].

□

Theorem 2.2.10. *The cup product gives a well-defined product in the cohomology ring* $H^*(X)$.

Proof. Let $[\phi] = [\phi'] \in C^k(X, \mathbb{Z})$ so that there exists $\phi'' \in C^{k-1}(X, \mathbb{Z})$ with $\delta^{k-1}(\phi'') = \phi - \phi'$. Let $[\psi] = [\psi'] \in C^\ell(X, \mathbb{Z})$ so that there exists $\psi'' \in C^{\ell-1}(X, \mathbb{Z})$ with $\delta^{\ell-1}(\psi'') = \psi - \psi'$. Then

$$\begin{aligned}
\phi \cup \psi - \phi' \cup \psi' &= \phi \cup \psi - \phi \cup \psi' + \phi \cup \psi' - \phi' \cup \psi' \\
&= \phi \cup (\psi - \psi') + (\phi - \phi') \cup \psi' \\
&= \phi \cup \delta^{\ell-1}(\psi'') + \delta^{k-1}(\phi'') \cup \psi' \\
&= (-1)^k \left(\delta^k(\phi) \cup \psi'' + (-1)^k \phi \cup \delta^{\ell-1}(\psi'')\right) + \qquad (4) \\
&\qquad \delta^{k-1}(\phi'') \cup \psi' + (-1)^{k-1} \phi'' \cup \delta^\ell(\psi') \qquad (5) \\
&= \delta^{k+\ell-1}((-1)^k \phi \cup \psi'' + \phi'' \cup \psi'),
\end{aligned}$$

where in lines (4) and (5) we use the fact that ϕ and ψ' are cycles, so $\delta^k(\phi) = 0$ and $\delta^\ell(\psi') = 0$. □

Cellular homology and cohomology is another interesting, perhaps more approachable way of thinking of these objects. We will illustrate cellular cohomology using an example of a specific type of compact complex manifold (see Munkres for the definition of a manifold [19]). Let X be a compact complex manifold such that $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$, where $X_k \setminus X_{k-1}$ is a disjoint union of b_k copies of $\mathbb{C}^k$. Let $Z_k^1, \ldots, Z_k^{b_k}$ be the closures of these copies of $\mathbb{C}^k$ in X. For example, Z_0^1 is a point with real dimension 0, Z_1^1 is a complex curve with real dimension 2, and Z_2^1 is a complex surface and has 4 real dimensions. Heading towards cohomology, we define $\overline{C}^{2k}(X)$ to be the free Abelian group generated by the set $\{Z_k^1, \ldots, Z_k^{b_k}\}$. These chains have a very interesting property, which stems from the relationship between the complex and real numbers. As the dimension of complex space increases by 1, we can think of it increasing in real dimension by 2, because every complex number has both a real and an imaginary component. Because of this, every other one of these chain groups $\overline{C}^k(X)$ is equal to zero. That is to say, $\overline{C}^{2k-1}(X) = 0$ and $\overline{C}^{2k+1}(X) = 0$, but (when $b_k > 0$) $\overline{C}^{2k}(X) \neq 0$ for $0 \leq 2k \leq 2n$. We then establish maps $\delta^k : \overline{C}^k(X) \to \overline{C}^{k+1}(X)$:

$$\begin{array}{ccccccccc} \to & \overline{C}^{2k-1} & \xrightarrow{\delta^{2k-1}} & \overline{C}^{2k} & \xrightarrow{\delta^{2k}} & \overline{C}^{2k+1} & \xrightarrow{\delta^{2k+1}} & \overline{C}^{2k+2} & \to \\ & \| & & & & \| & & & \\ & 0 & & & & 0 & & & \end{array} \tag{6}$$

All the maps in (6) are the zero map. The maps δ^{2k} send all elements of their domain to 0, and the others δ^{2k+1} are the identity map, simply taking zero to zero in the range. As we said earlier in Definition 2.2.5, the k-th cohomology of X is $H^k(X) = \frac{Ker(\delta^k)}{Im(\delta^{k-1})}$. The image of the map δ^{2k-1} is 0, and the kernel of δ^{2k} is everything in its domain $\overline{C}^{2k}$, so the cohomology is $H^{2k}(X) = \frac{Ker(\delta^{2k})}{Im(\delta^{2k-1})} = \frac{\overline{C}^{2k}}{0} = \overline{C}^{2k} \cong \mathbb{Z}^{b_k}$. The image of the map δ^{2k} is 0, and the kernel of δ^{2k+1} is everything in its domain $\overline{C}^{2k+1} = 0$, so the cohomology is $H^{2k+1}(X) = \frac{Ker(\delta^{2k+1})}{Im(\delta^{2k})} = \frac{\overline{C}^{2k+1}}{0} = 0$.

Therefore, the k-th cohomology of the compact complex manifold X is:

$$H^k(X) = \left\{ \begin{array}{ll} \mathbb{Z}^{b_t} & \text{if } k = 2t \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{array} \right\}.$$

Example 2.2.11. Recall that $\mathbb{RP}^2 = \{[x : y : z] \in \mathbb{RP}^2 : z \neq 0\}(\cong \mathbb{R}^2) \cup \{[x : y : 0] \in \mathbb{RP}^2 : y \neq 0\}(\cong \mathbb{R}^1) \cup \{[x : 0 : 0] \in \mathbb{RP}^2 : x \neq 0\}(\cong \mathbb{R}^0)$. That is, $\mathbb{RP}^2$ has cellular decomposition $X_0 \subset X_1 \subset X_2 = \mathbb{RP}^2$, where $X_0 = \mathbb{R}^0$, $X_1 \setminus X_0 = \mathbb{R}^1$, and $X_2 \setminus X_1 = \mathbb{R}^2$. Similarly, $\mathbb{CP}^2$ has cellular decomposition $X_0 \subset X_1 \subset X_2 = \mathbb{CP}^2$, where $X_0 = \mathbb{C}^0$, $X_1 \setminus X_0 = \mathbb{C}^1$, and $X_2 \setminus X_1 = \mathbb{C}^2$. Using real cellular cohomology, we find $H^0(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}$, $H^1(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}$, $H^2(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}$, and all other cohomology groups are 0, so in complex cellular cohomology, $H^0(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}$, $H^2(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}$, $H^4(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}$, and the other groups are 0. In fact, using the cup product formula one can check that the generator of $H^0(\mathbb{CP}^2, \mathbb{Z})$ is the identity element in $H^*(\mathbb{CP}^2)$ and if

$[x]$ is the generator of $H^2(\mathbb{CP}^2, \mathbb{Z})$, then $H^4(\mathbb{CP}^2, \mathbb{Z})$ is generated by $[x]^2$. So

$$H^*(\mathbb{CP}^2) \cong \mathbb{Z}[x]/(x^3).$$

Similarly, $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1})$.

The Kunneth Formula allows us to compute the cohomology ring of a product space.

Theorem 2.2.12 (Kunneth Formula)**.** *If topological spaces X and Y both admit cellular decompositions and $H^k(Y)$ is a finitely-generated free $\mathbb{Z}$-module for all k, then $H^*(X \times Y) \cong H^*(X) \otimes_{\mathbb{Z}} H^*(Y)$.*

Proof. See Hatcher [12, Theorem 3.16].

□

Example 2.2.13. Applying the Kunneth Formula repeatedly,

$$H^*((\mathbb{P}^2)^\ell) \cong \frac{\mathbb{Z}[x_1, \dots, x_\ell]}{(x_1^3, \dots, x_\ell^3)}.$$

2.3 Moduli Spaces of Arrangements

In order to count the number of arrangements of a given intersection lattice type that pass through d points in general position, we introduce a *moduli space*, the space of all arrangements with fixed intersection lattice. In general, a moduli space is a set whose elements represent geometric objects of some fixed kind. Moduli spaces are often equipped with extra structure so that we can interpret this set as a geometric object in its own right. It is interesting to investigate the geometry of these moduli spaces (see Harris and Morrison [11]). For example, we might be interested in whether the moduli space is smooth (a manifold) or singular. If the moduli space is singular, the points in the singular set may represent interesting examples of our geometric objects. Also, as we will see, the dimension of the moduli space of arrangements with fixed intersection lattice is the number d such that our enumerative problem has a finite non-zero answer. A ready example of a moduli space is $\mathbb{P}^n$, because each point in $\mathbb{P}^n$ represents a line through the origin in $\mathbb{R}^{n+1}$. The dual space $\mathbb{P}^{2*}$ is also a moduli space: the elements of $\mathbb{P}^{2*}$ represent lines in $\mathbb{P}^2$. The product $(\mathbb{P}^{2*})^\ell$ is yet another example of a moduli space. Every point in the new space $(\mathbb{P}^{2*})^\ell$ represents a labeled arrangement of ℓ lines in $\mathbb{P}^2$.

To define the moduli space of arrangements we need to introduce some notation. Choose coordinates for our ambient projective space $\mathbb{P}^n$ so that the basis for $\mathbb{P}^{n*}$ is $\{x_0, \dots, x_n\}$. For

$v \in (\mathbb{P}^{n*})^\ell$ with $v = (V_1, \ldots, V_\ell)$ where $V_i = (v_{i0} : \cdots : v_{in})$ let the arrangement defined by v be

$$\mathcal{A}(v) = \bigcup_{i=1}^{\ell} \{v_{i0}x_0 + \cdots + v_{in}x_n = 0\}.$$

In the case where $n = 2$, elements $v = (V_1, \ldots, V_\ell)$ of $(\mathbb{P}^{2*})^\ell$ represent line arrangements of ℓ lines $V_1, V_2, \ldots, V_\ell$ with line equations $v_{i0}x_0 + v_{i1}x_1 + v_{i2}x_2 = 0$.

Definition 2.3.1. Fix a geometric lattice L of rank n with ℓ rank-1 elements. The moduli space of L is

$$\mathcal{M}(L) = \left\{ v \in (\mathbb{P}^{n*})^\ell | L(\mathcal{A}(v)) \cong L \right\}.$$

These moduli spaces $\mathcal{M}(L)$ are extremely interesting, very difficult to understand, and have wide-ranging applications throughout mathematics. For example, in [10] Hacking, Keel, and Tevelev continue the work started by Kapranov in [15] on the compactification of the moduli space of generic arrangements, and show this compactification can be constructed combinatorially. In [28], Terao presents a stratification of the moduli space $\mathcal{M}(L)$ in order to assist computations with logarithmic Gauss-Manin connections. In another direction, Yuzvinsky in [31] shows that the set of free arrangements in the moduli space $\mathcal{M}(L)$ is a Zariski open set.

In [22], Rimányi, Némethi, and László give a connection between some enumerative problems and the equivariant cohomology of the moduli space $\mathcal{M}(L)$. In particular, they show that the Zariski closure of $\mathcal{M}(L)$ as viewed as a subvariety of $(\mathbb{C}^{n+1})^\ell$ is not the variety given only by the solutions to the determinantal equations supplied by L. These determinantal equations are presented nicely in Terao [28]. We study in detail the case when L is the lattice of the braid arrangement A_4 in Section 6.3.

3 Counting Arrangements

The main question we address involves counting hyperplane arrangements that satisfy some geometric conditions.

Question 3.0.2. Fix an intersection lattice L and d points in general position in the projective plane $\mathbb{P}^2$. How many distinct line arrangements with intersection lattice isomorphic to L contain these d points?

Though these problems are easy to pose, they are very challenging to answer. We use a variety of methods to answer these questions, since no one method can be easily applied in every case.

As well, we apply several methods to some problems: getting the same answer using different approaches helps validate our results. The first, and most intuitive, method for answering these enumerative problems involves straightforward counting.

3.1 Combinatorics

Combinatorics is the branch of mathematics concerned with the study of finite or countable discrete structures. We make frequent use of the binomial coefficient $\binom{n}{k} = \frac{n!}{(n-k)!(k!)}$ in our combinatorial solutions. The binomial coefficient can be interpreted as the number of different ways to choose k things from a set of n things (without replacement). It is very important to remember that the binomial coefficient ignores the order in which items are picked.

We start with a simple intersection type: ℓ hyperplanes in $\mathbb{P}^2$ in general position. Because we are working in $\mathbb{P}^2$, our hyperplane arrangements are line arrangements. The lines are in general position if no three intersect in a point.

Example 3.1.1. How many arrangements of four lines in general position are there through d points in $\mathbb{P}^2$ that lie in general position?

One way to fix the location of a line in the arrangement is to insist that the line pases through two given points. Let $[x_0 : y_0 : z_0]$ and $[x_1 : y_1 : z_1]$ be two distinct points. Then

$$\left\{ \begin{array}{rcl} ax_0 + by_0 + cz_0 & = & 0 \\ ax_1 + by_1 + cz_1 & = & 0 \end{array} \right\}$$

is a rank-2 system of linear equations and so has a 1-dimensional solution space. Thus there is a nonzero vector $[a : b : c]$ that is the solution to this system of equations and represents the line between points $[x_0 : y_0 : z_0]$ and $[x_1 : y_1 : z_1]$. If we only have one given point, there is an infinite family of lines through that point, each with a different slope. Since there are four lines and each line is determined by two points in general position, it seems reasonable that we must fix $d = 8$ points in general position in order to obtain a finite number of arrangements of this intersection lattice type through the d points. In Theorem 3.1.2 we'll give an explicit proof that $d = 8$.

Now that we know $d = 8$, we can begin to calculate the number of different arrangements there are through the same eight points in general position. We do this by using combinatorics. There are $\binom{8}{2} = \frac{8!}{6!2!} = 28$ ways to pick a pair of two points from the collection of eight points. These two points determine the first line. Then we choose one of the $\binom{6}{2} = \frac{6!}{4!2!} = 15$ pairs of two points from the remaining six points for the second line. Next, we fix the third line by putting it through one of the $\binom{4}{2} = 6$ pairs that are left over. Finally, the last line goes through the only two remaining points. You might guess that this means there are $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 28 * 15 * 6 * 1 = 2520$ different arrangements of four generic lines through eight points in general position, but this is incorrect. Consider the arrangements of four lines in general position in Figure 8.

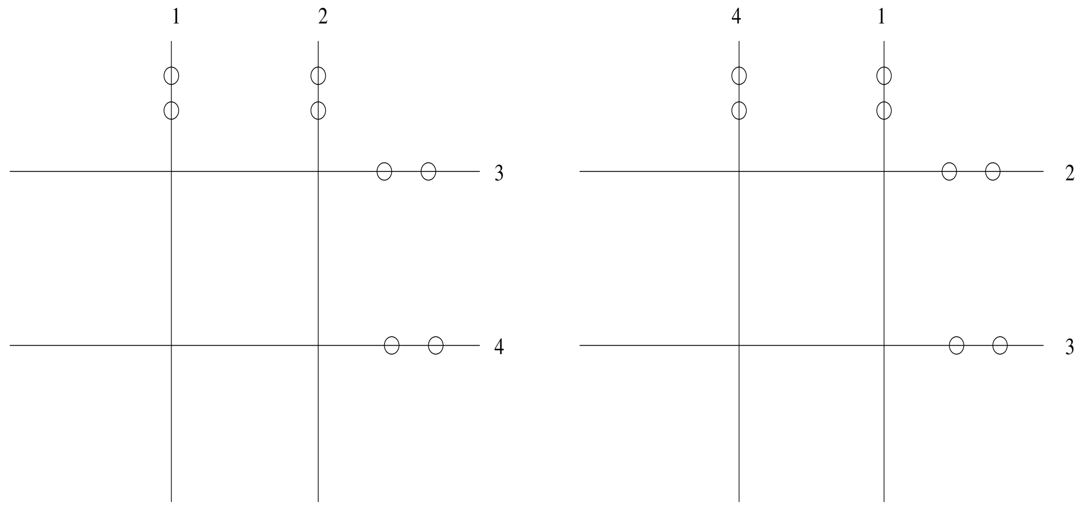

Figure 8: Two identical arrangements with different line labels

These are the same arrangement, but the lines in the arrangement were picked in a different order, i.e. the labels on the lines are different. This means that some of the arrangements we counted in our 2520 result are identical copies. In fact, each arrangement is being counted $4! = 24$ times, because there are $4!$ ways to permute the labels on the four lines. We call this factor of 24 "overcounting." The accurate answer to this question, then, is that there are $\frac{2520}{24} = 105$ different arrangements of four generic lines through eight fixed points in general position.

We use these same methods to count the number of generic arrangements of ℓ lines through d points.

Theorem 3.1.2. *Given $d = 2\ell$ points in general position, there are*

$$\frac{(2\ell)!}{\ell! 2^{\ell}}$$

generic arrangements of ℓ lines that pass through all the points. When $d < 2\ell$, the count is ∞, and when $d > 2\ell$, the count is 0.

Proof. Let $\mathcal{A}$ be an arrangement of ℓ generic lines.

We begin by proving that d, the number of points, must be equal to 2ℓ for the count to be a nonzero finite number. Because d is the number of points, it must be a non-zero positive integer. Let $d > 2\ell$ points be fixed in general position in $\mathbb{P}^2$. By definition of general position, no three of these points are collinear. However, because there are ℓ lines in the arrangement and each of the general points must lie on at least one of these lines, the pigeonhole principle implies that at least one of the lines contains 3 or more points. This violates the condition that these points are in general position. Therefore, the greatest number of points in general position that can possibly be contained by ℓ lines is 2ℓ, so $d \leq 2\ell$. Now, let $d < 2\ell$ points be fixed in general position in $\mathbb{P}^2$. Label these points $p_1, p_2, \ldots, p_d$. Suppose there are an even number of points $d = 2k$. Begin placing lines through pairs of points. Using our labels, we can fix line ℓ_1 through points p_1 and p_2, line ℓ_2 through p_3 and p_4, etc. Because there are fewer than 2ℓ of these points in general position, we will only be able to fix $k < \ell$ lines. This means that, as long as they don't pass through the intersection of two other lines in the arrangement, $\ell - k$ lines can be placed anywhere in the projective plane to create a suitable generic arrangement. Therefore, the number of generic arrangements through an even number of points $d < 2\ell$ is infinite. Suppose now that there are an odd number of points $d = 2k + 1$, labeled from 1 to $2k + 1$. Following the same construction as above, we fix k lines through the first $2k$ points. This leaves us with $\ell - k$ lines to place in the arrangement. One of these remaining lines must pass through the last point in general position, but there are an infinite number of distinct arrangements that can be made by varying the slope of this line. If $\ell - k$ is greater than 1, then those other $\ell - k - 1$ lines can be arranged in an infinite number of ways that still result in a generic arrangement of ℓ lines. Therefore, $d \geq 2\ell$. Now we've proven $2\ell \geq d \geq 2\ell \Rightarrow d = 2\ell$.

The process of Example 3.1.1 in which we fix every line in the arrangement (picking two points at a time from however many are left) is the same for any number of lines ℓ. This gives us a factor of $\binom{2\ell}{2}\binom{2\ell-2}{2}\binom{2\ell-4}{2}\cdots\binom{4}{2}\binom{2}{2} = \frac{(2\ell)!(2\ell-2)!(2\ell-4)!\cdots(4)!(2)!}{(2\ell-2)!2!(2\ell-4)!2!(2\ell-6)!2!\cdots2!2!2!0!} = \frac{(2\ell)!}{2^\ell}$. Just as in Example 3.1.1, we must adjust this number for the overcount. Because any line can be assigned any of the ℓ labels, the overcount is $\ell!$.

Therefore, the number of arrangements of ℓ generic lines through 2ℓ points is

$$\frac{\frac{(2\ell)!}{2^\ell}}{\ell!} = \frac{(2\ell)!}{\ell! 2^\ell}.$$

□

Another combinatorial intersection type in $\mathbb{P}^2$ that admits nice enumerative counts is arrangements in which all lines intersect in a single point. We call arrangements of this type "pencils."

Example 3.1.3. How many points d must be placed in general position for there to be a finite nonzero number of pencils of four lines through these d points? How many different pencils of four lines pass through these d points?

We give a heuristic argument to determine the number of points d in this problem. A rigorous argument will be given in Theorem 3.1.4. We begin in much the same way we did for the generic arrangement. Fix $d = 6$ points in general position. Place each of the first two lines in the arrangement through two of the points in general position. Because we are in projective space, these two lines must intersect. The final two lines of the arrangement each pass through one of the remaining two points in general position and the intersection of the first two lines. This ensures that all four lines intersect at a single point, which is required for the arrangement to be a pencil. Our choices of points in general position to place each of the lines (2 of the 6 points for the first line, 2 of the remaining 4 points for the second line, etc.) are reflected in the combinatorics: $\binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{1}{1} = \frac{6!}{2^2} = 180$. Determining the overcount for the pencil is not as straightforward as for generic arrangements.

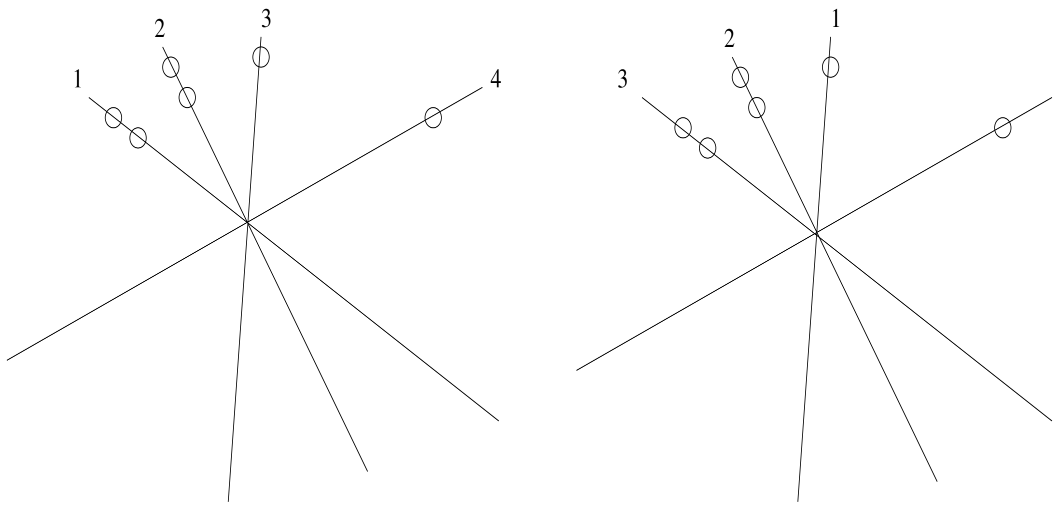

Figure 9: Different labels on a pencil of four lines

Whereas for generic arrangements any change in labeling results in an identical arrangement, this is not the case for the pencil. In Figure 9 lines 1 and 3 are determined by a different number of points, making them geometrically different, so their labels can not be interchanged. However, the labels on 1 and 2, or 3 and 4, may be switched without affecting the arrangement. Therefore, the overcount for this arrangement is $2 * 2 = 4$. This means that the number of pencils of four lines through six points in general position is $\frac{180}{4} = 45$.

Theorem 3.1.4. *Given $d = \ell + 2$ points in general position, there are*

$$\frac{1}{2}\binom{\ell+2}{2}\binom{\ell}{2}$$

arrangements of pencils of ℓ lines through all d points. If $d > \ell + 2$ then there are no pencils through the d points, and if $d < \ell + 2$ then there are infinitely man such pencils.

Proof. Let $\mathcal{A}$ be an arrangement of ℓ lines in a pencil.

First we must prove that $d = \ell + 2$ points must be placed in general position in $\mathbb{P}^2$ to fix a non-zero finite number of pencils of ℓ lines. Assume that $d > \ell+2$. By the pigeonhole principle, at least three of the ℓ lines contain two or more points in general position. Apart from preventing three points from being collinear, general position imposes more restrictions: no three lines through pairs of points can intersect in a common point in space. This presents us with a contradiction. The ℓ lines in $\mathcal{A}$ are arranged in a pencil, so they all intersect at a single point, but three of the lines in the arrangement each go through 2 points in general position, making it impossible for them to intersect at a point. Therefore, $d \leq \ell + 2$. Now let $d < \ell + 2$, and label these points $p_1, p_2, \ldots, p_d$. Fix the first line ℓ_1 through points p_1 and p_2 and the second line ℓ_2 through points p_3 and p_4. These lines intersect at a point which we will call p. We have $\ell - 2$ lines left to fix, but only $d - 4 \leq \ell - 2$ points in general position. We can place $d - 4$ of the $\ell - 2$ remaining lines such that each of them passes through one point in general position and the intersection point p of ℓ_1 and ℓ_2; however, we will be left with at least one line. This line must contain the point p, but can be any of an infinite family of lines that all result in an arrangement of our desired intersection lattice type – a pencil. Therefore, $d \geq \ell + 2$. Because the number of arrangements is 0 if $d > \ell + 2$ and ∞ if $d < \ell + 2$, you only get a non-zero finite number of pencils of ℓ lines if you fix $d = \ell + 2$ points in general position.

Construct the arrangement $\mathcal{A}$ in the same manner as in Example 3.1.3. Before dividing by the symmetry group, our count is $\binom{\ell+2}{2}\binom{\ell}{2}\binom{\ell-2}{1}\binom{\ell-3}{1}\cdots\binom{3}{1}\binom{2}{1}\binom{1}{1}$. Recall that the labels on the first two lines are interchangeable, because they each pass through two general points, as are the labels on the remaining $\ell - 2$ lines, because they pass through one general point. Accordingly, the size of the symmetry group is $2 * (\ell - 2)!$.

The resulting count is

$$\frac{\binom{\ell+2}{2}\binom{\ell}{2}\binom{\ell-2}{1}\binom{\ell-3}{1}\cdots\binom{3}{1}\binom{2}{1}\binom{1}{1}}{2(\ell-2)!}$$

$$= \frac{\binom{\ell+2}{2}\binom{\ell}{2}\binom{\ell-2}{1}\binom{\ell-3}{1}\cdots\binom{3}{1}\binom{2}{1}\binom{1}{1}}{2(\ell-2)!}$$

$$= \frac{\binom{\ell+2}{2}\binom{\ell}{2}(\ell-2)!}{2(\ell-2)!} = \frac{\binom{\ell+2}{2}\binom{\ell}{2}}{2}.$$

$\square$

Another intersection lattice type we have investigated is the tie-fighter, so called because of its resemblance to the Star Wars spaceship. The tie-fighter is an arrangement of five lines, as shown in Figure 10.

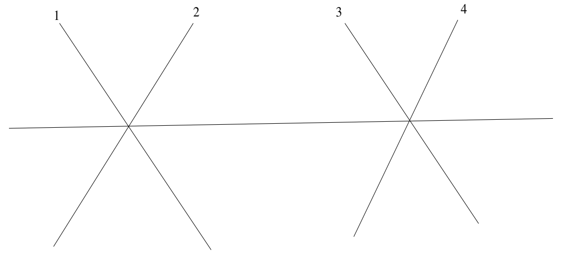

Figure 10: A "tie-fighter" arrangement of five lines

We use the same kind of reasoning as for the pencil and generic arrangements to determine that we need $d = 8$ points in general position to get a finite nonzero number of tie-fighter arrangements. The tie-fighter is an important intersection lattice to study because it presents a number of the common challenges we face in solving our enumerative problem. Whereas the generic arrangement and pencil are constructed using a single methodology, there are multiple ways of building tie-fighter arrangements, and these must all be accounted for. The first construction of the tie-fighter is similar to that of generic arrangements. We place the first four lines by choosing pairs of points in general position, resulting in four generic lines. The final line in the arrangement passes through the intersection of the first two generic lines and the intersection of the third and fourth generic lines. There are eight different labelings on the first four lines that give an arrangement with equivalent intersection lattice. The labels on generic lines L_1 and L_2 may be switched, as may the labels on L_3 and L_4, contributing 4 to the size of the symmetry group. This number is then doubled if we switch both L_1 and L_2 for L_3 and L_4. If we don't keep the pairs of labels together, then the arrangement will change after relabeling. This brings our final symmetry group size to 8. The resulting combinatorial count is straightforward: $\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{8} = \frac{8!}{2^7} = 315$. As we said earlier, there are other constructions for the tie-fighter. In one of these other constructions, two lines are initially placed through 4 points in general position. The next line passes through one of the remaining general points and the intersection of the first two lines. The fourth line is another generic line, fixed by choosing two of the last three points in general position. The final line goes through the last general point and the intersection of the third and fourth line. This construction is depicted in Figure 11.

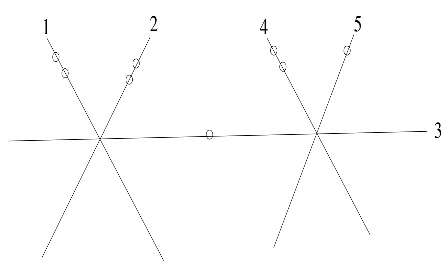

Figure 11: A construction of the tie-fighter arrangement

The labels on the first two generic lines can be exchanged without changing the arrangement, but none of the others can. This is the only relabeling that maintains both the intersection lattice and the number of points on each line. Lines L_1 and L_2 both pass through two points in general position, whereas line L_4 passes through two and line L_5 passes through one. Because L_1 and L_2 must each pass through two points in general position, we can not exchange their labels with line L_4 and L_5. Therefore, the count for this construction is $\frac{\binom{8}{2}\binom{6}{2}}{2}\binom{4}{1}\binom{3}{2}\binom{1}{1} = \frac{8!}{2^4} = 2520$. There is yet another construction for the tie-fighter, which is similar to the previous one in that there are three generic lines, but in this particular construction the line containing both triple points passes through two points in general position, instead of one. We begin by placing a generic line through two points in general position. This line will eventually contain both triple points in the arrangement. Next, choose two more general points and place another generic line through them. The third line connects the intersection of the first two lines with one of the four remaining points in general position. The process used to place the second and third lines is repeated for the fourth and fifth, creating the other triple point on the first line. As far as overcounting is concerned, the only relabeling that preserves both the intersection lattice and the number of points on each line exchanges the labels on lines L_2 and L_3 for those on lines L_4 and L_5. Done in this way, relabeling doesn't alter the arrangement. In this case, the count comes out to be $\frac{\binom{8}{2}\binom{6}{2}\binom{4}{1}\binom{3}{2}\binom{1}{1}}{2} = \frac{8!}{2^4} = 2520$. After careful consideration, we find that there are no other constructions, leaving us with the three constructions displayed in Figure 12.

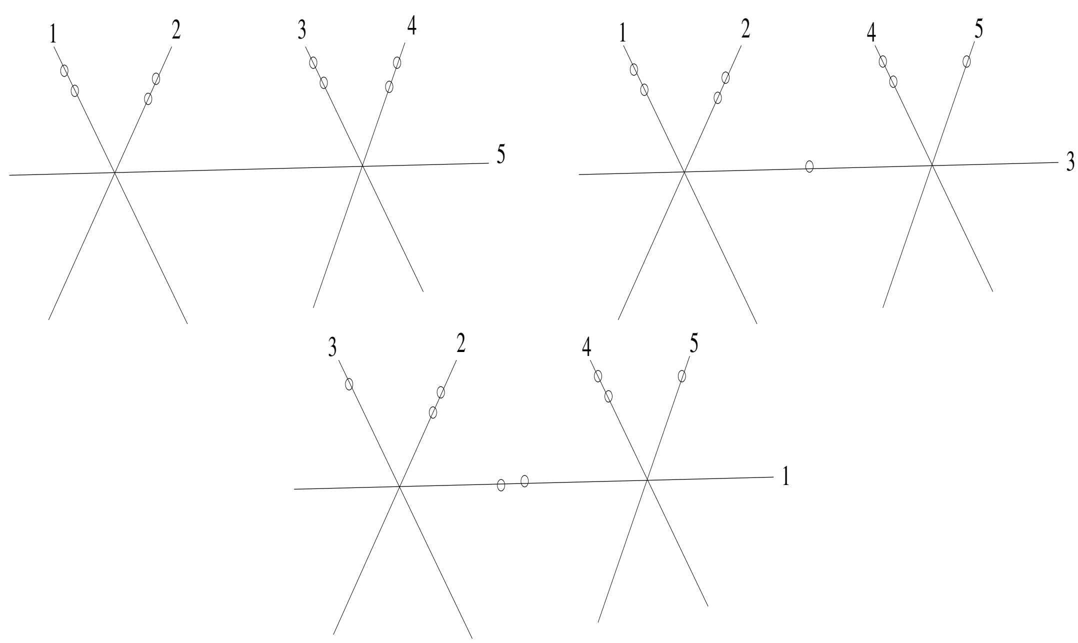

Figure 12: The three constructions of the tie-fighter arrangement

Therefore, the final answer to our enumerative problem for the tie-fighter arrangement is $315 + 2520 + 2520 = 5355$.

4 Cohomology and the Chow ring

In our setting, we will make use of a particular property of cohomology to equate certain cohomology classes. This is a complicated topic that is discussed in detail in Katz chapter 5 [16]. Here we just sketch the ideas. Given a topological space X, we have the projection map $\pi_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$ sending (x, p) to p. We identify the preimages $\pi_2^{-1}(p)$ with X. Given two points a and b in $\mathbb{P}^1$, any closed subspace S of $X \times \mathbb{P}^1$ induces a deformation from $V_a = S \cap \pi_2^{-1}(a)$ to $V_b = S \cap \pi_2^{-1}(b)$. Fix a basis in $\mathbb{P}^1$ identified as $\mathbb{R} \cup \infty$ such that $a = 0$ and $b = \infty$. The boundary of $S \cap \{X \times [0, \infty]\}$ is zero in cohomology. On the other hand, the boundary is also $[V_a] - [V_b]$. So $[V_a] - [V_b] = 0$ and $[V_a] = [V_b]$. By this reasoning, if we can deform any variety V_1 to another variety V_2, then $[V_1] = [V_2]$.

For an explicit example, let $X = \mathbb{P}^2$ and consider $S = \mathbb{V}((s - t)z^2 + (t)xy)$ in $\mathbb{P}^2 \times \mathbb{P}^1$. Choosing a basis for $\mathbb{P}^1$ so that $[s : t] = [0 : 1]$ describes the point 0 in $\mathbb{P}^1$ and $[s : t] = [1 : 0]$ describes the point ∞ in $\mathbb{P}^1$, $V_0 = \mathbb{V}(xy - z^2)$ – the set of points in $\mathbb{P}^2$ where $xy - z^2 = 0$ – and $V_\infty = \mathbb{V}(z^2)$. Therefore, the cohomology class $[\mathbb{V}(z^2 - xy)]$ is equal to the class $[\mathbb{V}(z^2)]$, which is $2[\mathbb{V}(z)]$. In general, the class of a degree d hypersurface in $\mathbb{P}^n$ equals $d[H]$ where $[H]$ is the class of a hyperplane H in $\mathbb{P}^n$.

Let's investigate the equivalence classes in cohomology a little more closely. Let H be a hyperplane in $\mathbb{P}^{2*}$. H has an equation $AX + BY + CZ = 0$ where $X, Y,$ and Z are the homogeneous coordinates on $\mathbb{P}^{2*}$. $[X : Y : Z]$ represents the line $Xx + Yy + Zz = 0$ in $\mathbb{R}^3$. In $\mathbb{P}^{2*}$ the coordinates $X, Y,$ and Z can take any real value as long as $AX + BY + CZ = 0$ for some fixed $A, B,$ and C and not all X,Y, and Z are zero. Then, once a particular set of values $X, Y,$ and Z is chosen, $x, y,$ and z may also take any real values such that $Xx + Yy + Zz = 0$. Therefore a hyperplane in $\mathbb{P}^{2*}$ consists of all lines through a fixed point in $\mathbb{P}^2$ (as in Example 2.1.4). Now let $H_1 = \{[X : Y : Z] : A_1X + B_1Y + C_1Z = 0\}$ and $H_2 = \{[X : Y : Z] : A_2X + B_2Y + C_2Z = 0\}$ be two hyperplanes in $\mathbb{P}^2$. These hypersurfaces are the boundaries of $S \subset \mathbb{P}^2 \times \mathbb{P}^1$ given by

$$(t - s)(A_1X + B_1Y + C_1Z) + s(A_2X + B_2Y + C_2Z) = 0$$

in the sense that $\pi_2^{-1}([0 : 1]) = H_1$ and $\pi_2^{-1}([1 : 0]) = H_2$. This shows, as we said earlier about the classes of boundaries in cohomology, $[H_1] = [H_2] = [H]$. The fact that the equivalence classes of all of these hyperplanes are equal is very important. It says that any hyperplane is equivalent to any other hyperplane it can be deformed to. In particular the class of the lines in $\mathbb{P}^2$ passing through a given point is well-defined and does not depend on the point.

The *Chow ring* $A_*(X)$ is another important cohomological object we make use of. Much like we did for singular cohomology, we must first define some relevant terms. For proofs and further explanations of these terms beyond the scope of this paper, see Fulton [7].

Definition 4.0.5. Let X be a topological space. The k-cycles $Z_k(X)$ of X is the free Abelain group on the k-dimensional subvarities of X.

$$Z_k(X) = \left\{ \sum n_i v_i : n_i \in \mathbb{Z} \text{ and } \dim(v_i) = k \right\}.$$

We say that $\alpha = \sum n_i v_i \in Z_k(X)$ is rationally equivalent to 0 if there exist finitely many dimension $k+1$ subvarieties w_j and nonconstant maps $\phi_j : w_j \to \mathbb{P}^1$ so that $\sum_j (\phi_i^{-1}([0:1]) - \phi_i^{-1}([1:0])) = \alpha$. Define $Rat_k(X) = \{\alpha \in \mathbb{Z}_k(X) : \alpha \text{ rationally equivalent to } 0\}$, a subgroup of $Z_k(X)$, and the k-th Chow group $A_k(X) = Z_k(X)/Rat_k(X)$.

Then two k-cycles α_1 and α_2 are equivalent if $\alpha_1 - \alpha_2$ is rationally equivalent to 0, i.e. $\alpha_1 - \alpha_2 \in Rat_k(X)$. Then $\oplus A_k(X) = A_*$ is a ring with product defined by $[v_1][v_2] = [v_1 \cap v_2]$, whenever v_1 and v_2 intersect transversely (that is, whenever the tangent spaces to v_1 and v_2 only meet in a space of codimension equal to codim(v_1)+codim(v_2)). The sum in A_k is defined by $[v_1]+[v_2] = [v_1 \cup v_2]$ when v_1 and v_2 intersect transversely. Moreover, when X is a smooth manifold, as is the case with $\mathbb{P}^2$, there is an isomorphism

$$A_*(X) \to H^*(X)$$

of graded rings sending $A_k(X)$ to $H^{n-k}(X)$, where $n = \dim(X)$.

Example 4.0.6. The Chow ring $A_*(\mathbb{CP}^n)$ is isomorphic to the cohomology ring $H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$. To be precise, the image of the class of a hyperplane in $\mathbb{CP}^n$ is the class $[x]$, the generator of $H^1(\mathbb{CP}^n)$. Thus,

$$A_*(\mathbb{CP}^n) \cong \frac{\mathbb{Z}[[H]]}{[H]^{n+1}} \cong \bigoplus_{0 \le t \le n} \mathbb{Z}[H]^t,$$

where $\mathbb{Z}[[H]]$ is the polynomial ring generated by $[H]$.

The Chow ring shares a nice property with the usual cohomology ring. Given a degree-d hypersurface $F = 0$ in $\mathbb{P}^n$, the class of the hypersurface $[F = 0]$ is equivalent to $d[H]$, where H is a hyperplane in $\mathbb{P}^n$. To see this, define a map $\phi : \mathbb{P}^n \to \mathbb{P}^1$ by $\phi(\mathbf{x}) = [F(\mathbf{x}) : L(\mathbf{x})^d]$, where $L = 0$ is the equation of a hyperplane. Then $\phi^{-1}([0:1]) = \{F = 0\}$ and $\phi^{-1}([1:0]) = \{L^d = 0\}$. So $\{F = 0\}$ is rationally equivalent to $\{L^d = 0\}$, i.e. $[F] = [L^d] = d[H]$. If $F_1 = 0, \ldots, F_n = 0$ are hypersurfaces of degrees $d_1, \ldots, d_n$ in $\mathbb{P}^n$ that intersect transversely (that is, at each point of intersection p the tangent spaces to the hypersurfaces intersect only at p), then

$$\begin{aligned} [\{F_0 = 0 \cap F_1 = 0 \cap \cdots \cap F_n = 0\}] &= [F_0 = 0][F_1 = 0] \ldots [F_n = 0] \\ &= (d_1[H])(d_2[H]), \ldots, (d_n[H]) \\ &= d_1 d_2 \ldots d_n [H]^n. \end{aligned}$$

Since n hyperplanes in $\mathbb{CP}^n$ intersect transversely in a point, $[H]^n$ represents a point in $\mathbb{CP}^n$, so we expect these hypersurface to intersect in $d_1 d_2 \ldots d_n$ points. This is the content of Bézout's Theorem.

Theorem 4.0.7 (Bézout's Theorem)**.** *If n hypersurfaces $F_1 = 0, \ldots, F_n = 0$ of degrees $d_1, \ldots, d_n$ in $\mathbb{P}^n$ intersect transversely, then the hypersurfaces intersect in $d_1 d_2 \cdots d_n$ points.*

This theorem is critical to enumerative results, because it allows us to count intersections by computing in the cohomology ring.

Recall that $\mathbb{P}^{2*}$ is the moduli space of lines in $\mathbb{P}^2$ and the product $(\mathbb{P}^{2*})^\ell$ is the moduli space of labeled arrangements of ℓ lines in $\mathbb{P}^2$. Since $\mathbb{P}^{2*}$ is isomorphic to $\mathbb{P}^2$, their cohomology rings are also isomorphic. Thus, using Example 2.2.13, the Chow ring of $(\mathbb{P}^{2*})^\ell$ is

$$A_*((\mathbb{P}^{2*})^\ell) = \frac{\mathbb{Z}[x_1, x_2, \ldots, x_\ell]}{(x_1^3, x_2^3, \ldots, x_\ell^3)},$$

where x_i corresponds to the class of a hyperplane in the i^{th} copy of $\mathbb{P}^{2*}$.

4.1 The Intersection Polynomial

The *intersection polynomial* of an arrangement is an element of the Chow ring. The intersection polynomial is of the form $(x_1 + x_2 + \cdots + x_\ell)^d * P_L(x_1, x_2, \ldots, x_\ell)$, where $P_L(x_1, x_2, \ldots, x_\ell)$ is a polynomial based on the intersection lattice type $L = L(\mathcal{A})$ of an arrangement $\mathcal{A}$ of ℓ hyperplanes through d points. The reason the intersection polynomial is so useful to us is that the coefficient of the top class of the polynomial $x_1^2 x_2^2 \cdots x_\ell^2$ represents the number of *labeled* arrangements with our lattice type and satisfying the geometric conditions. After computing this number, we can divide by the size of the appropriate symmetry group to obtain our desired count. Since the quotient of the Chow ring is generated by $(x_1^3, x_2^3, \ldots, x_\ell^3)$, there is no term in the intersection polynomial containing a variable raised to the third (or higher) power. Therefore, the term in which every x_i is at its maximum degree is $x_1^2 x_2^2 \cdots x_\ell^2$. As we said earlier, these x_i are classes in the Chow ring of $(\mathbb{P}^{2*})^\ell$, so they each represent the class of a hyperplane in the i^{th} copy of $\mathbb{P}^{2*}$. The product of two such classes is the class of a point in the i^{th} copy of $\mathbb{P}^{2*}$. The term $x_1^2 x_2^2 \cdots x_\ell^2$ corresponds to imposing restrictions on each factor in $(\mathbb{P}^{2*})^\ell$ so that the resulting class represents a point in the product $(\mathbb{P}^{2*})^\ell$, that is, a labeled arrangement of ℓ lines. To obtain the number of unlabeled arrangements, we simply divide this number by the size of the symmetry group, which can be computed with the symmetries program found in Appendix A.

We must be very careful in determining $P_L(x_1, x_2, \ldots, x_\ell)$. This factor in the polynomial corresponds to the multi-intersections (triple points, quadruple points, etc.) in the arrangement and is vital to obtaining an accurate count. Let A be an arrangement of ℓ hyperplanes in $\mathbb{P}^2$. Note that if lines L_1, L_2, and L_3 intersect in a common point, then they each represent planes in $\mathbb{C}^3$ that intersect in a common line. If the equation defining line L_i is given by $a_i x + b_i y + c_i z = 0$ (for

$i \in \{1, 2, 3\}$), then the planes share a common line if and only if the matrix

$$M_{1,2,3} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

has a determinant equal to zero. This determinant equation is a degree-3 polynomial in the nine variables $a_1, a_2, a_3, b_1, \ldots, c_3$. In fact, it is a multihomogeneous polynomial in the sense that each term in the polynomial involves one variable with index 1, one variable with index 2, and one variable of index 3. The zero set of such a polynomial is rationally equivalent to the zero set of any of its terms, for example, $a_1b_2c_3$. The zero set of such a polynomial is a union of three hyperplanes ($a_1 = 0$, $b_2 = 0$, and $c_3 = 0$). Since the class of a union is the sum of the classes in the Chow ring, the class of such a hypersurface in the Chow ring is $x_1+x_2+x_3$. Similarly, if distinct lines i, j, and k meet in a triple point, then the corresponding class of this geometric condition is $x_i + x_j + x_k$.

To ensure that an arrangement passes through a fixed point $p = [x_0 : y_0 : z_0]$ we need one of the lines in the arrangement ($a_ix + b_iy + c_iz = 0$ for $i \in \{1, 2, \ldots, \ell\}$) to pass through p. This is equivalent to requiring that

$$\prod_{1\leq i\leq \ell} (a_ix_0 + b_iy_0 + c_iz_0) = 0.$$

This is a multihomogeneous polynomial as well, so the zero set of the polynomial is rationally equivalent to the zero set of any of its terms, such as $a_1a_2 \ldots a_\ell$. In this case, the zero set is a union of ℓ hyperplanes ($a_1 = 0$,$a_2 = 0$, $\ldots$, $a_\ell = 0$). The class of such a hypersurface in the Chow ring is $x_1+x_2+\cdots+x_\ell$. Imposing d independent point conditions produces the term $(x_1+x_2+\cdots+x_\ell)^d$ in the intersection polynomial for an arrangement.

To better illustrate this point, we look at a few examples, for which we already know the count from combinatorics.

Example 4.1.1. Let $\mathcal{A}$ be a generic arrangement of 4 lines through 8 points in general position in $\mathbb{P}^2$. There are no triple points in $\mathcal{A}$, so there are no determinantal conditions that must be accounted for in the intersection polynomial. Therefore, the intersection polynomial of this intersection lattice is $(x_1 + x_2 + x_3 + x_4)^8$. Using SAGE, we take the partial derivative of this polynomial twice with respect to each x_i, and then divide by $2^\ell = 2^4$ to determine that the coefficient of $x_1^2x_2^2x_3^2x_4^2$ is 2520. As we already know, the size of the symmetry group for generic arrangements is $\ell!$, so we divided 2520 by $4!$ to determine that there are $\frac{2520}{4!} = 105$ generic arrangements of 4 lines through 8 points in general position in $\mathbb{P}^2$. This matches up exactly with the number we obtained using combinatorics.

Example 4.1.2. Let $\mathcal{A}$ be a pencil of 4 lines through 6 points in general position in $\mathbb{P}^2$. By virtue of the intersection lattice, all 4 lines in $\mathcal{A}$ meet at a single point, so there are $\binom{4}{3}$ possible collections of three lines that we could say meet in a point. However, we don't need that many determinantal conditions to force the lines to meet in a quadruple point. It is sufficient to say, for example, that L_1, L_2, and L_3 meet in a triple point, and L_2, L_3, and L_4 meet in a triple point, because then all other combinations of three lines are forced to intersect at a single point. Therefore, any two such collections of three lines may be used for the determinantal conditions. Using the two collections I suggested earlier, we can create the intersection polynomial for this intersection lattice: $(x_1+x_2+x_3)(x_2+x_3+x_4)(x_1+x_2+x_3+x_4)^6$. We take the derivative of this polynomial in the same manner as before to find that the coefficient of $x_1^2x_2^2x_3^2x_4^2$ is 1440. At this point, it is tempting to divide by the size of the symmetry group ($4!$) and claim that there are 60 pencils of 4 lines through 6 points in general position; however, as we know from combinatorics, this isn't true. This is an example of why we must be careful in our interpretation of results from the intersection polynomial. The coefficient of $x_1^2x_2^2x_3^2x_4^2$ counts all arrangements that contain d points and satisfy the determinantal conditions imposed by the intersection polynomial, not just the arrangements with a particular intersection lattice. In this case, arrangements made up of two equal lines and two generic lines are also included in the intersection polynomial. If $L_2 = L_3$, then we still satisfy the two tripe-points conditions in the intersection polynomial because $\det(M_{1,2,3}) = 0$, but our arrangement is no longer a pencil, because L_1 and L_4 do not meet on $L_2 = L_3$, as depicted in Figure 13.

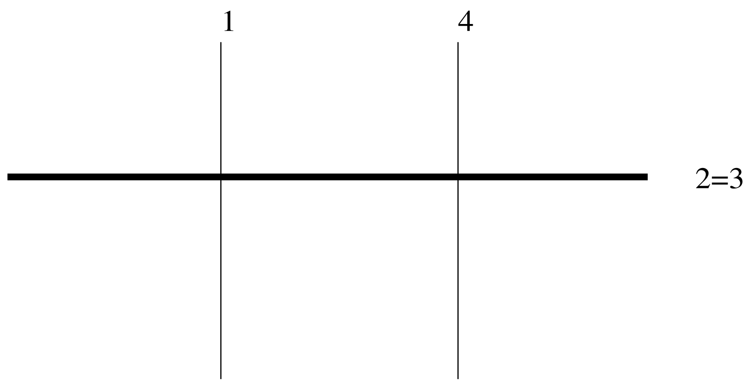

Figure 13: Arrangement with a double line $L_2 = L_3$

Therefore, we must calculate the number of such arrangements, and take into account any multiplicity these arrangements have in $(\mathbb{P}^{2*})^\ell$. There are $\binom{6}{2}\binom{4}{2}\binom{2}{2} = 90$ double-line arrangements that must be removed, each of which counts with multiplicity 4. The computation for determining the multiplicity of an arrangement is conducted by a complicated computer algorithm whose contents are otherwise irrelevant to this project. See Fulton [6, Chapter 1] for theoretical details and Smith and Sturmfels [23] for ways to implement the computation in Macaulay2. This means that the accurate number of labeled pencils of 4 lines is $1440 - 4(90) = 1080$. Dividing 1080 by the number of relabelings ($4! = 24$), we get $\frac{1080}{24} = 45$ unlabeled pencils of 4 lines, the same count we calculated using combinatorics.

5 The Tutte Polynomial

The Tutte polynomial is a multivariate polynomial that encodes the dependent sets of a matroid (see Sokal [25]). The equation for the Tutte polynomial of a hyperplane arrangement with matroid M, ground set E, and number of lines $\ell = |E|$ is

$$\tilde{Z}_M(q, \bar{v}) = \sum_{A \subseteq E} (q^{-r(A)}) \prod_{j=1}^{k} (v_j)$$

, where $\bar{v} = (v_1, v_2, \dots, v_\ell)$, $A = \{\ell_1, \ell_2, \dots, \ell_k\} \subseteq E$, and $r(A) = \operatorname{codim}(\ell_1 \cap \ell_2 \cap \cdots \cap \ell_k)$. We have found that it is more convenient to work with a slight variation of the Tutte polynomial,

$$Z_M(q, \bar{v}) = \tilde{Z}_M(q^{-1}, \bar{v}),$$

which for simplicity we just call the Tutte polynomial.

For example, consider the arrangement of three lines in general position. The Tutte polynomial of the intersection lattice of this arrangement is

$$Z_M(q, \bar{v}) = 1 + q(v_1 + v_2 + v_3) + q^2(v_1v_2 + v_1v_3 + v_2v_3) + q^3(v_1v_2v_3).$$

A quick glance at this polynomial can tell you the intersection information of our arrangement. The factors of any terms with a coefficient of q^1 intersect in a line (in $\mathbb{P}^2$, a codimension 1 object), those with a coefficient of q^2 in a point (in $\mathbb{P}^2$, a codimension 2 object), and those with a coefficient of q^3 intersect in the empty set (in $\mathbb{P}^2$, a codimension 3 object), which means they don't intersect at all. In this case, our Tutte polynomial exactly reflects the intersections of the arrangement, because every combination of two lines intersects at a point, but all three lines do not intersect.

Since the Tutte polynomial of the intersection lattice of an arrangement $\mathcal{A}$ contains all the intersection data of $\mathcal{A}$, it seems plausible that the polynomial could be used to compute the number of arrangements going through d points in general position, where d is chosen appropriately so that the answer is finite and nonzero. One possibility is that this count equals the evaluation of $Z_M(q, \bar{v})$ at a suitable q and $\bar{v}$. When $q = 1$, the Tutte polynomial simplifies nicely to $Z_M(1, \bar{v}) = \prod_{e \in E}(1 + v_e)$. This means that if we find a regular pattern in the factors of an arrangement count for a changing number of lines, we can find the values of $\bar{v}$ with $q = 1$. If you look at the number of unlabeled arrangements in generic position, you see that the count is $15 = 1 \times 3 \times 5$ for $\ell = 3$, $105 = 1 \times 3 \times 5 \times 7$ for $\ell = 4$, $945 = 1 \times 3 \times 5 \times 7 \times 9$ for $\ell = 5$, and so on. The pattern here is, for ℓ lines, the count is $1 \times 3 \times 5 \times \cdots \times (2\ell - 1) = (2\ell - 1)!!$.

Theorem 5.0.3. *Let $\mathcal{A}$ be a generic arrangement of ℓ lines through 2ℓ points in general position in $\mathbb{P}^2$. The number of arrangements passing through these 2ℓ points in general position is equal to*

$$Z_M(1, (0, 2, 4, \dots, 2\ell - 2)) = (2\ell - 1)!!.$$

Proof. Let $\mathcal{A}$ be an arrangement satisfying the conditions in Theorem 5.0.3. As we proved in Theorem 3.1.2, the number of such arrangements is equal to

$$\begin{aligned}\frac{(2\ell)!}{\ell!2^{\ell}} &= \frac{(2\ell)(2\ell-1)(2\ell-2)(2\ell-3)\cdots(3)(2)(1)}{2^{\ell}(\ell)(\ell-1)(\ell-2)(\ell-3)\cdots(3)(2)(1)}\\ &= \frac{(2\ell)(2\ell-1)(2\ell-2)(2\ell-3)\cdots(3)(2)(1)}{(2\ell)(2\ell-2)(2\ell-4)(2\ell-6)\cdots(6)(4)(2)}\\ &= (2\ell-1)(2\ell-3)\cdots(5)(3)(1) = (2\ell-1)!!.\end{aligned}$$

Let $\bar{v} = (0, 2, 4, \ldots, 2\ell-2)$. Evaluated at $\bar{v}$, the Tutte polynomial is

$$\begin{aligned}Z_M(1, \bar{v}) &= (1+0)(1+2)(1+4)\cdots(1+(2\ell-2))\\ &= 1(3)(5)\cdots(2\ell-1) = (2\ell-1)!!.\end{aligned}$$

Therefore, the number of generic arrangements of ℓ lines through 2ℓ points in general position is equal to $Z_M(1, (0, 2, 4, \ldots, 2\ell-2))$. ☐

We have also analyzed the Tutte polynomials of pencils of lines.

Theorem 5.0.4. *Let $\mathcal{A}$ be a pencil of ℓ lines through $\ell+2$ points in general position in $\mathbb{P}^2$. Then the number of distinct arrangements $\mathcal{A}$ through the $\ell+2$ points is equal to $(k+1)(2k+1)$, where*

$$k = \left\{ \begin{array}{ll} \frac{\ell^2+\ell-4}{4} & \textit{if } \ell = 0 \textit{ mod } 4 \textit{ or } 3 \textit{ mod } 4 \\ -\frac{\ell^2+\ell+2}{4} & \textit{if } \ell = 1 \textit{ mod } 4 \textit{ or } 2 \textit{ mod } 4 \end{array} \right\}$$

Proof. Fix the intersection type of our arrangement $\mathcal{A}$ of ℓ lines to be the pencil. From Theorem 3.1.4, we know there are $\frac{\binom{\ell+2}{2}\binom{\ell}{2}}{2}$ distinct arrangements with this intersection type through $\ell+2$ points in general position. Let the number of lines ℓ be equal to $4n$ or $4n+3$ for some non-negative integer n. Then the proposed count from Theorem 5.0.4 is

$$\begin{aligned}&(\frac{\ell^2+\ell-4}{4}+1)(2(\frac{\ell^2+\ell-4}{4})+1) = (\frac{\ell^2+\ell}{4})(\frac{\ell^2+\ell-2}{2})\\ &= \frac{(\ell+1)(\ell)(\ell+2)(\ell-1)}{8} = \frac{(\ell+2)!(\ell)!}{2(\ell!)2\left((\ell-2)!\right)(2)} = \frac{\binom{\ell+2}{2}\binom{\ell}{2}}{2}.\end{aligned}$$

Now let the number of lines be equal to $4n+1$ or $4n+2$ for some non-negative integer n. By Theorem 5.0.4, the count is

$$\begin{aligned}&(-\frac{\ell^2+\ell+2}{4}+1)(2(-\frac{\ell^2+\ell+2}{4})+1) = (\frac{-\ell^2-\ell+2}{4})(\frac{-2\ell^2-2\ell}{4})\\ &= (\frac{\ell^2+\ell-2}{4})(\frac{\ell^2+\ell}{2}) = \frac{\binom{\ell+2}{2}\binom{\ell}{2}}{2}.\end{aligned}$$

☐

We have found that using the Tutte polynomial to calculate the count for other lattice types is more difficult, because no pattern is apparent. We have written a computer program in Sage that takes a matrix, the columns of which represent the coefficients of line equations for the arrangement, calculates the independent sets, and outputs the Tutte polynomial for the intersection type. We have also written a program that takes the Tutte polynomial, the arrangement count, and bounds for q and the v_i, and outputs all solutions to the polynomial equal to the count within the given bounds. These functions can be found in Appendix A. This program may be able to provide some insight into the connection between the finite, nonzero number of arrangements of a given lattice intersection type and that intersection type's Tutte polynomial, but as of this writing we have been unable to find such a precise formula.

6 The Braid Arrangement

The main result of this paper is solving the enumerative problem for a very interesting but complicated intersection lattice type - the braid arrangement.

Definition 6.0.5. Let V be an n-dimensional vector space, and let

$$H_{ij} = \{(v_1, v_2, \ldots, v_n) \in V | v_i = v_j\}$$

for $1 \leq i < j \leq n$. We define the *braid arrangement* to be

$$A_n = \{H_{ij} | 1 \leq i < j \leq n\}.$$

We focus our attention on an arrangement related to A_4, which we will also call the "braid arrangement" in this paper. To obtain this arrangement, note that the line $L = \{(v_1, v_2, v_3, v_4) : v_1 = v_2 = v_3 = v_4\}$ is contained in each hyperplane $H_{ij} \in A_4$. Moding out this line (i.e. intersecting with the hyperplane $v_1 + v_2 + v_3 + v_4 = 0$) gives an arrangement in $\mathbb{C}^3$ which determines the braid arrangement in $\mathbb{P}^2$. Figure 14 on the next page illustrates two example of braid arrangements, when viewed in the projective plane $\mathbb{P}^2$. As we can see, the braid arrangement in $\mathbb{P}^2$ contains four triple points, with each of the six lines containing two triple points.

As with the solutions to all of our enumerative problems, the first step is to find how many points d must be fixed in general position in order to get a finite nonzero number of arrangements. We claim that this number is 8.

Theorem 6.0.6. *There is a finite nonzero number of braid arrangements through 8 points in general position in $\mathbb{P}^2$.*

Proof. Let $d < 8$. Recall our proof that the number of points that must be fixed in general position to get a finite nonzero number of generic arrangements must be 2ℓ. In that situation, if $d < 2\ell$, we

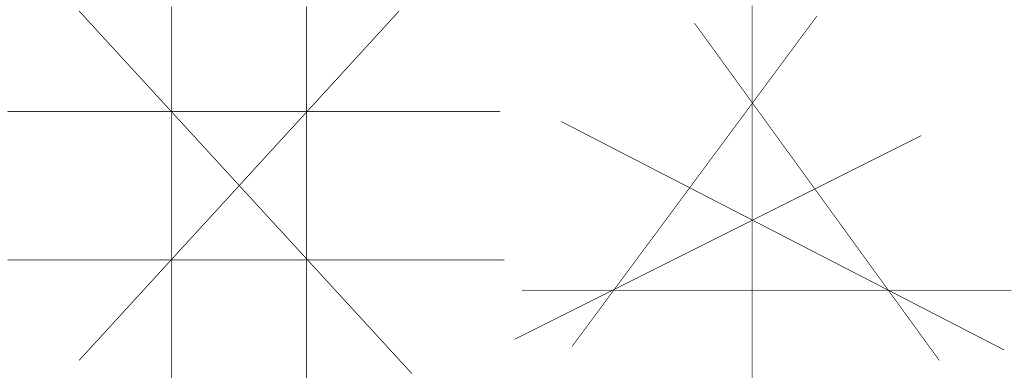

Figure 14: Two arrangements with the lattice type of the braid arrangement of 6 hyperplanes in $\mathbb{P}^2$.

get an infinite number of possible generic arrangements. This is analogous to the case of the braid arrangement when $d < 8$, because four of the lines in the braid arrangement are in general position. In our standard picture of the braid arrangement, these are the first four lines that are placed. When $d < 8$, we can place these first four lines in an infinite number of ways, so when we add the last two lines (joining pairs of the $\binom{4}{2} = 6$ double points to produce 4 triple points) we get an infinite number of possible braid arrangements. Therefore, when $d < 8$ the number of braid arrangements is infinite. Now, let $d > 8$. We can fix the first four lines without issue through 8 of the points in general position, with some number of points left over. If there are more than 4 points remaining, we get 0 possible braid arrangements, because there is no way the last two lines can each contain more than two points in general position. If there are 4 points remaining, the resulting arrangement after placing the last two lines is a generic arrangement, not a braid arrangement. In the case with 3 left over points, one of the last two lines must contain two by the pigeon hole principle. This means that we have five lines in general position. The braid arrangement, by definition, has four triple points in it. This means that the last line would have pass through the final point and four of the intersections of the five generic lines. This is impossible, so the number of braid arrangements is 0. Assume now that we have 2 points for the last two lines. If one of the lines contains both points, we are able to use the same argument as above to say that the final line cant pass through four intersection points. If instead each line contains one of the points in general position, we would still have to be able to put a line through a general point and two of the intersection points of the four generic lines. Because the points are in general position, this will never occur, so the number of arrangements is again 0. We are left at last with the case $d = 9$. One of the last two lines can be placed through two of the intersections of the four generic lines, creating two triple points. At this point we are confronted with the same problem as before: putting a line through a point in general position and two intersection points of generic lines. As we said earlier, this cannot be done by definition of general position, making it impossible to obtain any braid arrangements. This means that when $d > 8$, there are 0 braid arrangements that can be created. Therefore, to get a finite nonzero number of braid arrangements, we must fix 8 points in general position in $\mathbb{P}^2$. $\square$

Now that we know the number of points required, and therefore the dimension of the moduli space, we can begin to solve the enumerative problem in earnest. To do this, we use both combinatorics and the intersection polynomial, and then compare the results.

6.1 Intersection Polynomial

The construction of the intersection polynomial for the braid arrangement is relatively simple. First, we assign labels to the lines.

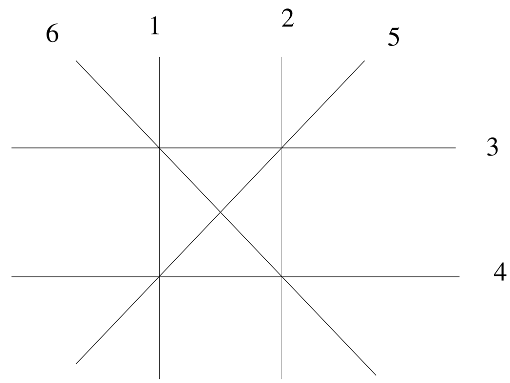

Figure 15: Braid Arrangement with hyperplanes labeled

In the braid arrangement, we wish to impose four triple point conditions, and the corresponding class of all four geometric conditions is just the product $(x_1 + x_3 + x_6)(x_2 + x_3 + x_5)(x_2 + x_4 + x_6)(x_1 + x_4 + x_5)$. As well, to ensure that the braid arrangement passes through a fixed point $p = [x_0 : y_0 : z_0]$ we need one of the six lines ($a_i x + b_i y + c_i z = 0$ for $i \in \{1, 2, \ldots, 6\}$) to pass through p. This is equivalent to requiring that

$$\prod_{1 \leq i \leq 6} (a_i x_0 + b_i y_0 + c_i z_0) = 0.$$

Again this is a multihomogeneous polynomial. The zero set of such a polynomial is rationally equivalent to the zero set of any of its terms, for example, $a_1 a_2 \ldots a_6$. The zero set of this kind of polynomial is a union of six hyperplanes ($a_1 = 0$,$a_2 = 0$, …, $a_6 = 0$). Since the class of a union is the sum of the classes in the Chow ring, the class of such a hypersurface in the Chow ring is $x_1 + x_2 + \cdots + x_6$. Imposing 8 independent point conditions produces the term $(x_1 + x_2 + \cdots + x_6)^8$ in the intersection polynomial for the braid arrangement.

The intersection polynomial for the braid arrangement is thus

$$(x_1 + x_3 + x_6)(x_2 + x_3 + x_5)(x_2 + x_4 + x_6)(x_1 + x_4 + x_5)(x_1 + x_2 + \cdots + x_6)^8.$$

Now we find the coefficient of $x_1^2x_2^2x_3^2x_4^2x_5^2x_6^2$, which we do by taking the double derivative of the polynomial with respect to each x_i and then divide by 2^6. The reason we take this particular derivative is because $x_1^2x_2^2x_3^2x_4^2x_5^2x_6^2$ is the only term in the polynomial that will not become 0 after taking all of these derivatives. In all of the other terms, one of the six x_i will have an exponent less than 2, and when the second derivative of that variable is taken that term will become 0. We divide by 2^6 because as you take the derivatives with respect to each of the six variables, the exponent drops down and increases the coefficient by a factor of two. After taking the derivatives, we get that the coefficient of $x_1^2x_2^2x_3^2x_4^2x_5^2x_6^2$ is $\frac{35320320}{2^6} = 551880$. We also need to calculate the symmetry group of the intersection polynomial. Using the symmetry group function described in Appendix A, we find that there are 24 different permutations of the line labels that preserve the intersection polynomial. While there are $6!$ possible ways to label 6 lines, only 24 of these labelings ensure that the collections of three lines comprising the four triple points remain the same. If we labeled the lines with one of the other permutations, we could go from having triple points $(L_1, L_3, L_6), (L_2, L_3, L_5), (L_2, L_4, L_6), (L_1, L_4, L_5)$ to $(L_1, L_4, L_6), (L_2, L_3, L_4), (L_1, L_3, L_5), (L_2, L_5, L_6)$, so the permutation is not a member of the symmetry group. So, we get $\frac{551880}{24} = 22995$ unlabeled arrangements. We expect to get the number of unlabeled arrangements (as opposed to labeled) because we divided by the number of possible different labelings, which eliminates overcounting. In this case, however, 22995 is *not* the number of unlabeled braid arrangements. There is another intersection type that satisfies the criteria imposed by the intersection polynomial, and we must subtract the number of arrangements of that type. The pencil of six lines is dimension 8, because for the pencil $d = \ell + 2$, and each of the triple point conditions (det($M_{1,3,6}$) =det($M_{2,3,5}$) =det($M_{2,4,6}$) =det($M_{1,4,5}$) $= 0$) is met, because all lines in a pencil intersect at a single point. This is one of the dangerous parts of using the intersection polynomial: it encodes some intersection information that must be true about the arrangement, such as det($M_{i,j,k}$) $= 0$, but doesn't necessarily include all of that information (e.g. it doesn't encode det($M_{i,j,k}$) $\neq 0$).

We must be careful to ensure that all extraneous lattice types contained in the count (in this case, 551880) are removed. For an intersection lattice type to be contained in this count, it must contain 6 lines, have dimension 8, and meet all of the triple point conditions set in the intersection polynomial. This does not necessarily mean that arrangements of this intersection type contain four triple points. As we see with the pencil, as long as the appropriate collections of three lines meet in a point, they are contained in the intersection polynomial count. Another important property of the braid arrangement that must be met by other arrangements, if they are included in the count, is that every line in the arrangement contains the intersections of two distinct pairs of lines. These intersection points may or may not be coincident. Comparing these criteria against the list of 10 possible intersection lattice types of 6 lines, which we have generated explicitly, the only types that satisfy all of them are the braid arrangement and the pencil. Therefore, the pencil of 6 lines is the only extraneous lattice type that is included in the count from the intersection polynomial, so we must subtract the number of labeled 6-pencils from 551880.

As stated in Theorem 3.1.4, the number of unlabeled arrangements of a pencil of ℓ lines is

$\frac{(\ell+2)!}{2^3(\ell-2)!}$, so for six lines there are $\frac{8!}{2^3(4!)} = 210$ unlabeled arrangements. We want to know the number of labeled arrangements, because 551880 was the number of labeled braid arrangements. To get this, we multiply 210 by $6!$, the number of different ways to label six lines. After doing this, we get $210 * 720 = 151200$. So the actual number of labeled braid arrangements is $551880 - 151200 = 400680$, which leads us to our main result.

Theorem 6.1.1 (Main Result)**.** *The number of unlabeled braid arrangements through* 8 *points in* $\mathbb{P}^2$ *is* $\frac{400680}{24} = 16695$.

This problem is a good illustration of how to use the intersection polynomial, and demonstrates some its potential pitfalls. Despite some of these small problems, we will see in the next section that the intersection polynomial is far cleaner than the combinatorial methods used to verify 16695.

6.2 Combinatorics

Just as we saw in the tie-fighter example, it is necessary to determine all of the various ways to construct an arrangement to ensure an accurate combinatorial count. With the braid arrangement, this is even more difficult. First, we looked at how many ways there are to partition 8 points (the dimension of the braid arrangement) into 6 bins (each representing the number of points on a particular line). There are three ways to do this: $[2, 2, 2, 2, 0, 0]$, $[2, 2, 2, 1, 1, 0]$, $[2, 2, 1, 1, 1, 1]$. Finding arrangements of the first two types is relatively straightforward:

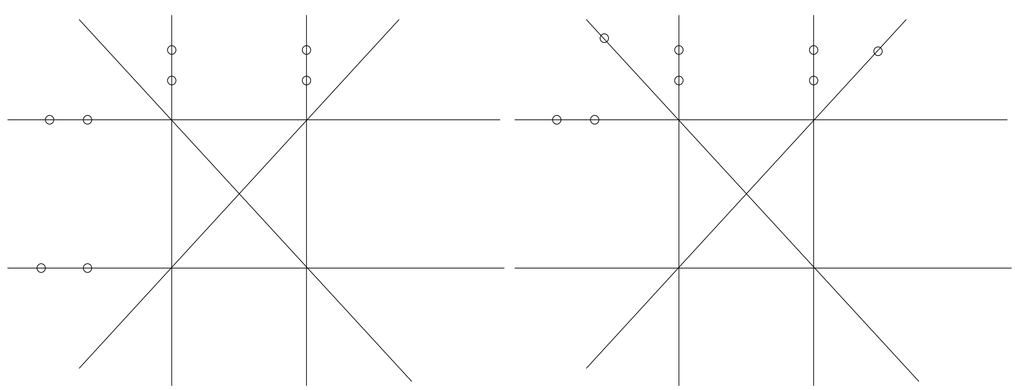

Figure 16: Braid Arrangements, $[2, 2, 2, 2, 0, 0]$ on left and $[2, 2, 2, 1, 1, 0]$ on right

The count for the $[2, 2, 2, 2, 0, 0]$ arrangements, illustrated on the left in Figure 16, is constructed using similar methods to arrangements in general position. The four lines through two points are generic (so their labels can be swapped), and the final two lines are chosen through pairs of double points created by the four generic lines (the labels on these two lines may be exchanged too), making the count

$$\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}\binom{3}{1}\binom{2}{1}}{4!2} = 315.$$

To compute the number of $[2,2,2,1,1,0]$ arrangements, on the right, we first place three generic lines through two points each, contributing a factor of $\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{3!}$. The next two lines each go through one point in general position and a double point of lines in the arrangement. The final line then passes through two double points of the arrangement. There are a number of different ways to do this. For example, when placing the fifth line, we want to account for cases in which the double point it goes through is an intersection of one of the three original generic lines, as well as cases in which the double point was created after placing the fourth line. After drawing out all possible distinct ways to place the last three lines, which was a painstaking process, we determined that the final three lines contribute a factor of 24 to the count, after removing overcounting. Therefore, the number of $[2,2,2,1,1,0]$ braid arrangements is

$$\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{3!} * 24 = 10080.$$

The final category of construction for the braid arrangement is $[2,2,1,1,1,1]$. This case introduced to us for the first time the possibility of having a line that, when first placed in the arrangement, doesn't have a fixed slope. Instead, we choose one of the generic points and assign to that point a line with a variable slope. After we've finished constructing the arrangement, we vary the slope of this line until we get an arrangement that is of the desired intersection type. Sometimes there is only one slope of the line that gives the appropriate intersection type, but other times there is more than one that does, or slopes that make arrangements of other interesting intersection types, like a pencil. To visualize these cases, we have made use of Geogebra [14], which allows us to vary the slope of the line and watch how the intersection type of the arrangement changes. When constructing arrangements with a variable-slope line, we label this line s to distinguish it from the others. Figure 17 at the top of the next page shows an example of a $[2,2,1,1,1,1]$ braid arrangement with a variable-slope line s, before and after finding the correct value for s.

The first step to constructing this arrangement is to make two generic lines through two points each, and then make a third line through one point and the intersection of the first two lines. The lines in the figure above are labeled in this order. Next, the line with variable slope s is placed through one of the remaining three points. The fifth line is through one of the last two points and one of the three double points in the arrangement (the intersections of the variable-slope line and the three previous lines) and the sixth line through one of the four resulting double points. Then, we vary s until the three lines around A in the figure above meet at a triple point. The labels on the first two lines are interchangeable, for an overcount of two, as are the label on the last three lines, for an overcount of $3! = 6$. Also, there are only two possible combinations of the the last four double points that result in a braid arrangement, so the last line only counts for a factor of 2, not 4. Therefore the count for this type of arrangement is $\frac{\binom{8}{2}\binom{6}{2}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{3}{1}\binom{2}{1}}{2(3!)} = 5040$.

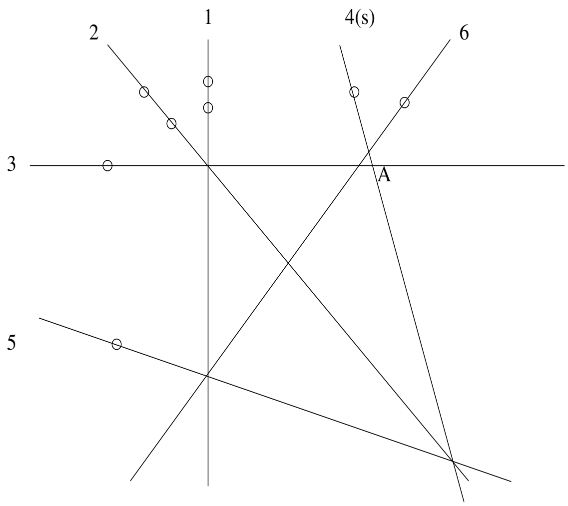

Figure 17: Constructing a braid arrangement with a variable-slope line by making A a triple point

There is another way to construct a $[2, 2, 1, 1, 1, 1]$ braid arrangement, with two variable-slope lines s and t, which itself breaks down into multiple different constructions. The multiple sub-constructions are due to the fact that braid arrangements can be obtained from having the fifth and sixth lines meet at a triple point, or not. To clarify, Figure 18 illustrates examples of each of these sub-constructions.

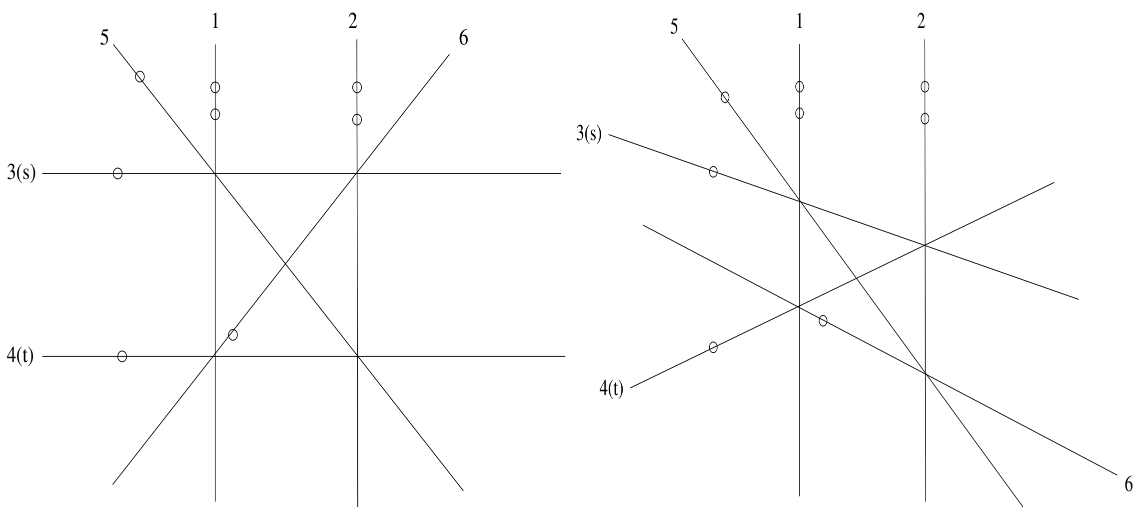

Figure 18: Constructing a braid arrangement with two variable-slope lines

In the figure on the left, lines 5 and 6 intersect at a double point, but in the figure on the right, they meet line 2 in a triple point. As before, the labels on the first two lines are interchangeable, but in this case all of the last four lines can be picked in any order, giving them an overcount of $4! = 24$. All together, this two variable-slope construction has the following count: $\frac{\binom{8}{2}\binom{6}{2}\binom{4}{1}\binom{3}{1}[\binom{2}{1}\binom{2}{1}+\binom{2}{1}\binom{2}{1}[1+1]]}{2*4!} = 1260.$

This accounts for all of the cases for the braid arrangement, so our final combinatorial count is $315 + 10080 + 5040 + 1260 = 16695$. This matches the count we got using intersection theory, so we are confident that this is the number of braid arrangements of six lines through eight points in general position.

6.3 Moduli Spaces of Braid Arrangements

Rimányi et al. [22] investigated the moduli space of braid arrangements as follows.

Let A be an $n \times \ell$ matrix. The number of columns of A,ℓ, is the number of lines in our arrangement, and the number of rows of A,n, is the dimension of the ambient space. A represents one particular arrangement of ℓ lines, with each column giving the coefficients of a line in $\mathbb{P}^2$. Let r be a function that takes elements of the power set of $\{1, 2, \ldots, \ell\}$ to $\mathbb{Z}$. Let $X = \{A \in M_{n\times\ell} : rank(\text{subset } I \text{ of columns}) = r(I)\}$. Let C be a specific line arrangement $\{v_1, \ldots, v_\ell\}$. Then X_C denotes the moduli space of the arrangement C, $X_C \subseteq \mathbb{C}^{n\ell}$. $Y_C = \overline{X_C}$ is the closure of X_C in $\mathbb{C}^{n\ell}$. Let's look at an example - the braid arrangement.

$$C = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & -1 & 0 & -1 \end{bmatrix}$$

In this matrix, the columns represent the line equations of each of the six lines in the braid arrangements. We label the columns from 1 to 6 such that the corresponding lines $L_1, L_2, \ldots, L_6$ meet at the following triple points:

$$\{L_1L_2L_6\}, \{L_1L_3L_5\}, \{L_2L_3L_4\}, \{L_4L_5L_6\}.$$

Now we want to construct X_C. X_C satisfies all of the rank conditions that subsets of the columns of C must satisfy to be a braid arrangement. So in this example, $X_C = \{$ subset I of columns of $C | r(\emptyset) = 0, r(\{i\}) = 1, r(\{ij\}) = 2, r(\{1, 2, 6\}) = r(\{1, 3, 5\}) = r(\{2, 3, 4\}) = r(\{4, 5, 6\}) = 2, r(\text{other } \{ijk\}) = 3, r(\text{others}) = 3\}$. These conditions force any two lines to intersect in a point (obvious because we're in projective space), four triples of lines to intersect in the triple points, and any other combination of three or more lines to intersect in the empty set. The $r(\{ij\}) = 2$ condition is satisfied by showing that at least one 2×2 minor of every pair of lines is not equal to 0. This is the same with the 3×3 minors of subsets of three or more non-triple point columns. To ensure that the triple points have rank 2, we want the determinants of the 3×3 matrices $M_{1,2,6}$, $M_{1,3,5}$, $M_{2,3,4}$, and $M_{4,5,6}$ to be zero.

We call the set cut out exclusively by the degree 3 triple point conditions the "naive" closure or Y_{naive}. We create Y_{naive} by only specifying the determinant conditions which require a matrix to have determinant equal to 0. So for the braid arrangement, that is $Y_{naive} = \{M \in \mathbb{C}^{3(6)} | det(M_{1,2,6}) = det(M_{2,3,4}) = det(M_{4,5,6}) = det(M_{1,3,5}) = 0\}$. It is tempting to think that Y_C equals the closure of X_C, but it can be observed quickly that this is not the appropriate closure for the moduli space X_C of the braid arrangement. Pencils of six lines, as we have already remarked, also satisfy these conditions, so they are part of Y_C.

Rimányi et al. point out that Y_{naive} can be thought of as $Y_{C_M} \cup \{M \in \mathbb{C}^{3(6)} | r(M) <= 2\}$, where Y_{C_M} is the appropriate closure of the moduli space of the braid arrangement. Rimányi et al. continue to say that, using computer algebra, he obtained 19 minimal generators for the ideal I_{C_M} of Y_{C_M}. In addition to the four degree 3 polynomials that make up $I_{naive} = (det(M_{1,2,6}), det(M_{2,3,4}), det(M_{4,5,6}), det(M_{1,3,5}))$, Rimányi et al. claim I_{C_M} also contains three degree 5 polynomials and twelve degree 6 polynomials. They demonstrate how they arrive at these polynomials, with the degree 5 polynomials coming from taking the sum of three of the degree 6 polynomials, pulling out a common factor, and taking that variable to be at infinity in the projective plane. For example, here are the three degree 6 polynomials that Rimányi et al. use to make a degree 5 polynomial:

$$\frac{z_5/y_5 - z_1/y_1}{z_3/y_3 - z_5/y_5} \cdot \frac{z_4/y_4 - z_3/y_3}{z_2/y_2 - z_4/y_4} \cdot \frac{x_6/y_6 - x_2/y_2}{x_1/y_3 - x_6/y_5} = -1,$$

$$\frac{x_1/y_1 - x_2/y_2}{x_2/y_2 - x_6/y_6} \cdot \frac{z_6/y_6 - z_4/y_4}{z_4/y_4 - z_5/y_5} \cdot \frac{z_5/y_5 - z_3/y_3}{z_3/y_3 - z_1/y_1} = -1,$$

$$\frac{z_1/y_1 - z_2/y_2}{z_2/y_2 - z_6/y_6} \cdot \frac{z_6/y_6 - z_4/y_4}{z_4/y_4 - z_5/y_5} \cdot \frac{x_5/y_5 - x_3/y_3}{x_3/y_3 - x_1/y_1} = -1.$$

The sum of these three degree 6 polynomials is

$$y_4(-x_5y_1z_3z_6y_2 + x_5y_1z_3z_2y_6 + x_3y_1z_2z_1y_6 - x_5y_3z_2z_1y_6 + z_1y_5z_3x_6y_2-$$
$$z_1y_5z_3x_2y_6z_3y_5z_2x_6y_1 + z_5y_3z_2x_6y_1 + x_2y_1z_6z_3y_5 - x_2y_1z_6z_5y_3 - x_6y_2z_5z_1y_3+$$
$$x_2y_6z_5z_1y_3 - z_1y_2z_6x_3y_5 + z_1y_2z_6x_5y_3 + z_6y_2z_5x_3y_1 - z_2y_6z_5x_3y_1).$$

Earlier in their paper, Rimányi et al. say that "in the projective plane (x, y, z) we can choose the y coordinate to be at infinity" ([22, page 6]). Because of this, they remove the y_4 factor entirely, resulting in a degree 5 polynomial that they claim is one of the minimal generators of I_{C_M}.

This method raised a number of questions in our minds. First, why are those three particular degree 6 polynomials chosen to be added? There are twelve degree 6 polynomials and only three degree 5 ones, so that means at least three of the degree 6 polynomials won't be used at all to make a degree 5 polynomial. Also, when Rimányi takes the y variable to be at infinity, why can we ignore y_4 (treat it as if it were 1) but still include the other y_i? It seems that if we assume $y = 1$ then this "degree 5" polynomial is in fact a degree 3 polynomial. We studied these questions using the computer algebra SAGE.

We begin by creating a matrix with our 18 variables: [1]

[1] SAGE code is printed in `typewriter font` and output is printed in *italics*.

`M = matrix`$\{\{a_1, a_2, a_3, a_4, a_5, a_6\}$`,`$\{b_1, b_2, b_3, b_4, b_5, b_6\}$`,`$\{c_1, c_2, c_3, c_4, c_5, c_6\}\}$

Next we create the "naive" ideal generated solely by the four degree 3 polynomials, which correspond to the four triple points on the arrangement:

$D126$ =`det(matrix`$\{\{a_1, a_2, a_6\}$`,`$\{b_1, b_2, b_6\}$`,`$\{c_1, c_2, c_6\}\}$`),`
$D135$ =`det(matrix`$\{\{a_1, a_3, a_5\}$`,`$\{b_1, b_3, b_5\}$`,`$\{c_1, c_3, c_5\}\}$`),`
$D234$ =`det(matrix`$\{\{a_2, a_3, a_4\}$`,`$\{b_2, b_3, b_4\}$`,`$\{c_2, c_3, c_4\}\}$`),`
$D456$ =`det(matrix`$\{\{a_4, a_5, a_6\}$`,`$\{b_4, b_5, b_6\}$`,`$\{c_4, c_5, c_6\}\}$`),`
`J = ideal(`$D126, D135, D234, D456$`)`

J = ideal($-a_6b_2c_1 + a_2b_6c_1 + a_6b_1c_2 - a_1b_6c_2 - a_2b_1c_6 + a_1b_2c_6, -a_5b_3c_1 + a_3b_5c_1 + a_5b_1c_3 - a_1b_5c_3 - a_3b_1c_5 + a_1b_3c_5, -a_4b_3c_2 + a_3b_4c_2 + a_4b_2c_3 - a_2b_4c_3 - a_3b_2c_4 + a_2b_3c_4, -a_6b_5c_4 + a_5b_6c_4 + a_6b_4c_5 - a_4b_6c_5 - a_5b_4c_6 + a_4b_5c_6$*)*

We know this is the "naive" ideal because it includes many arrangements that are not braid arrangements. To remove these unwanted arrangements, we use the ideal quotient or "colon ideal." The zero set of the colon ideal can be interpreted as the set difference between two ideals (actually, the difference is not of sets, but of schemes [4], but this level of detail will not concern us here). We want to remove the pencils, so we create an ideal made up of all the 3×3 minors of M and then take the colon ideal of J with ideal of the minors.

At this point, we have an ideal generated by fourteen polynomials. There are the four degree 3 polynomials that we expect, as well as ten degree 6 polynomials.

`MM = minors(3,M)`

$F_1 = b_3b_5b_6c_1c_2c_4 - b_2b_5b_6c_1c_3c_4 - b_3b_4b_6c_1c_2c_5 + b_1b-4b_6c_2c_3c_5 + b_2b_3b_6c_1c_4c_5 - b_1b_3b_6c_2c_4c_5 + b_2b_4b_5c_1c_3c_6 - b_1b_4b_5c_2c_3c_6 - b_2b_3b_5c_1c_4c_6 + b_1b_2b_5c_3c_4c_6 + b_1b_3b_4c_2c_5c_6 - b_1b_2b_4c_3c_5c_6$

$F_2 = a_3b_5b_6c_1c_2c_4 - a_2b_5b_6c_1c_3c_4 - a_3b_4b_6c_1c_2c_5 - a_4b_2b_6c_1c_3c_5 + a_2b_4b_6c_1c_3c_5 + a_4b_1b_6c_2c_3c_5 + a_3b_2b_6c_1c_4c_5 - a_3b_1b_6c_2c_4c_5 + a_4b_2b_5c_1c_3c_6 - a_4b_1b_5c_2c_3c_6 - a_3b_2b_5c_1c_4c_6 + a_2b_1b_5c_3c_4c_6 + a_3b_1b_4c_2c_5c_6 - a_2b_1b_4c_3c_5c_6$

$F_3 = a_3a_5b_6c_1c_2c_4 - a_2a_5b_6c_1c_3c_4 - a_3a_4b_6c_1c_2c_5 + a_1a_4b_6c_2c_3c_5 + a_2a_3b_6c_1c_4c_5 - a_1a_3b_6c_2c_4c_5 + a_4a_5b_2c_1c_3c_6 - a_4a_5b_1c_2c_3c_6 - a_3a_5b_2c_1c_4c_6 + a_2a_5b_1c_3c_4c_6 + a_3a_4b_1c_2c_5c_6 - a_1a_4b_2c_3c_5c_6 - a_2a_3b_1c_4c_5c_6 + a_1a_3b_2c_4c_5c_6$

$F_4 = a_3a_5a_6c_1c_2c_4 - a_2a_5a_6c_1c_3c_4 - a_3a_4a_6c_1c_2c_5 + a_1a_4a_6c_2c_3c_5 + a_2a_3a_6c_1c_4c_5 - a_1a_3a_6c_2c_4c_5 + a_2a_4a_5c_1c_3c_6 - a_1a_4a_5c_2c_3c_6 - a_2a_3a_5c_1c_4c_6 + a_1a_2a_5c_3c_4c_6 + a_1a_3a_4c_2c_5c_6 - a_1a_2a_4c_3c_5c_6$

$F_5 = a_5b_1b_3b_6c_2c_4 - a_1b_3b_5b_6c_2c_4 - a_5b_1b_2b_6c_3c_4 + a_1b_2b_5b_6c_3c_4 - a_3b_1b_4b_6c_2c_5 + a_1b_3b_4b_6c_2c_5 + a_3b_1b_2b_6c_4c_5 - a_1b_2b_3b_6c_4c_5 - a_5b_1b_3b_4c_2c_6 + a_3b_1b_4b_5c_2c_6 + a_5b_1b_2b_4c_3c_6 - a_1b_2b_4b_5c_3c_6 - a_3b_1b_2b_5c_4c_6 + a_1b_2b_3b_5c_4c_6$

$F_6 = a_3a_5b_1b_6c_2c_4 - a_1a_3b_5b_6c_2c_4 - a_2a_5b_1b_6c_3c_4 + a_1a_2b_5b_6c_3c_4 - a_3a_4b_1b_6c_2c_5 + a_1a_3b_4b_6c_2c_5 + a_1a_4b_2b_6c_3c_5 - a_1a_2b_4b_6c_3c_5 + a_2a_3b_1b_6c_4c_5 - a_1a_3b_2b_6c_4c_5 - a_3a_5b_1b_4c_2c_6 + a_3a_4b_1b_5c_2c_6 + a_2a_5b_1b_4c_3c_6 - a_1a_4b_2b_5c_3c_6 - a_2a_3b_1b_5c_4c_6 + a_1a_3b_2b_5c_4c_6$

$F_7 = a_3a_5a_6b_1c_2c_4 - a_1a_3a_5b_6c_2c_4 - a_2a_5a_6b_1c_3c_4 + a_1a_2a_5b_6c_3c_4 - a_3a_4a_6b_1c_2c_5 + a_1a_3a_4b_6c_2c_5 + a_1a_4a_6b_2c_3c_5 - a_1a_2a_4b_6c_3c_5 + a_2a_3a_6b_1c_4c_5 - a_1a_3a_6b_2c_4c_5 + a_2a_4a_5b_1c_3c_6 - a_1a_4a_5b_2c_3c_6 - a_2a_3a_5b_1c_4c_6 + a_1a_3a_5b_2c_4c_6$
$F_8 = a_3a_5b_1b_2b_6c_4 - a_2a_5b_1b_3b_6c_4 - a_1a_3b_2b_5b_6c_4 + a_1a_2b_3b_5b_6c_4 - a_3a_4b_1b_2b_6c_5 + a_1a_4b_2b_3b_6c_5 + a_2a_3b_1b_4b_6c_5 - a_1a_2b_3b_4b_6c_5 - a_3a_5b_1b_2b_4c_6 + a_2a_5b_1b_3b_4c_6 + a_3a_4b_1b_2b_5c_6 - a_1a_4b_2b_3b_5c_6 - a_2a_3b_1b_4b_5c_6 + a_1a_3b_2b_4b_5c_6$
$F_9 = a_3a_5a_6b_1b_2c_4 - a_2a_5a_6b_1b_3c_4 - a_1a_3a_5b_2b_6c_4 + a_1a_2a_5b_3b_6c_4 - a_3a_4a_6b_1b_2c_5 + a_1a_4a_6b_2b_3c_5 + a_2a_3a_6b_1b_4c_5 - a_1a_3a_6b_2b_4c_5 + a_1a_3a_4b_2b_6c_5 - a_1a_2a_4b_3b_6c_5 + a_2a_4a_5b_1b_3c_6 - a_1a_4a_5b_2b_3c_6 - a_2a_3a_5b_1b_4c_6 + a_1a_3a_5b_2b_4c_6$
$F_{10} = a_3a_5a_6b_1b_2b_4 - a_2a_5a_6b_1b_3b_4 - a_3a_4a_6b_1b_2b_5 + a_1a_4a_6b_2b_3b_5 + a_2a_3a_6b_1b_4b_5 - a_1a_3a_6b_2b_4b_5 + a_2a_4a_5b_1b_3b_6 - a_1a_4a_5b_2b_3b_6 - a_2a_3a_5b_1b_4b_6 + a_1a_2a_5b_3b_4b_6 + a_1a_3a_4b_2b_5b_6 - a_1a_2a_4b_3b_5b_6$

`J:MM==ideal(`$F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}, D126, D135, D234, D456$`)`

true

There are some interesting things to note about the generators of `J:MM`. We already know that the degree 3 polynomials determine the triple points on the arrangement, but three of the degree 6 polynomials are different than the others. Those three generators only contain two of the three variable letters (i.e. only $a\&b$, $a\&c$, or $b\&c$, as opposed to all three). Also, these polynomials are "symmetric" with respect to their subscripts. If we look at the last generator F_{10} we see that the first two terms are $a_3a_5a_6b_1b_2b_4 - a_2a_5a_6b_1b_3b_4 - \dots$ and the last two are $\dots + a_1a_3a_4b_2b_5b_6 - a_1a_2a_4b_3b_5b_6$. The subscripts of a in the first two terms are the subscripts of b in the last two, and vice versa. All of the two-letter polynomials have 12 terms, with the letters and signs changing after the sixth term. We believe these polynomials are associated with the three double points on the arrangement, but are still unsure about the precise relationship.

Theoretically, our `J:MM` ideal should be equal to Rimányi et al.'s I_{C_M}. After all, he said earlier that $Y_{naive} = Y_{C_M} \cup \{M \in \mathbb{C}^{3x6} | r(M) <= 2\}$, and what we did with the colon ideal was remove $\{M \in \mathbb{C}^{3x6} | r(M) <= 2\}$ from Y_{naive}. In actuality, however, there appear to be some big differences. `J:MM` doesn't have any degree 5 generators and only has ten degree 6 generating polynomials. We decided to test the degree 5 polynomial that they explicitly stated was a minimal generator of I_{C_M}, which we saw earlier. It turns out that the degree 5 polynomial given by Rimányi et al. is an element of the ideal generated by our 14 polynomials $(F_1, F_2, \dots, F_{10}, D126, D135, D234, D456)$. This means that the list of polynomials that Rimányi et al. claim are the minimal defining equations for the braid arrangement contain some redundant polynomials (the three degree 5 and two of the degree 6), and therefore by definition can not be minimal defining equations.

6.4 Applications of the Braid Arrangement

The braid arrangement has applications in widely-varied fields, such as robotics and political theory.

Say that we have n robots in the real plane such that we do not want them to run into one another. Now view $\mathbb{R}^2$ as $\mathbb{C}$, and let $F_n(\mathbb{C})$ be the set of all possible positions of the robots such that they are not touching. Let $M(A_n) = V/ \cup H_{ij}$ be the complex complement of the braid arrangement. Then $F_n(\mathbb{C}) = M(A_n)$. Recall that the elements of the braid arrangement are of the form $\{(v_1, v_2, \ldots, v_n) \in V | v_i = v_j\}$. In this context, these points are locations on the plane at which some robots are colliding. In fact, the braid arrangements represent all of the undesired points, so if we remove the braid arrangements from V, we are left with all of the possible configurations of robots $\{(v_1, v_2, \ldots, v_n) \in V | v_i \neq v_j\}$. These points are exactly the elements of $F_n(\mathbb{C})$, the places at which no robots are touching. This is a way in which the braid arrangement can be used to improve robot motion planning. Yuzvinsky et al. [5] studied algorithms that move the robots from one configuration (point in $F_n(\mathbb{C})$) to another without ever passing through the braid arrangement, ensuring no collisions.

In the realm of political theory, the braid arrangement can be used to prove Arrow's Impossibility Theorem, which is far from obvious. Here is the situation: we are given m voters and n policy options to be ranked. The goal is to find a "good" way to decide the final ranking of the n options, based on the preferences of the m voters. We define a social welfare function to be a method to decide the final ranking given any possible combination of society's votes such that the following axioms are satisfied:

(i) If everyone prefers i to j then so does society, and

(ii) Whether society prefers i to j depends only on each individual's preferences for i and j and does not include their rankings for other policies

We call a social welfare function in which only one person's vote counts a dictatorship, and that person a dictator. With this in mind, we give Arrow's Impossibility Theorem:

Theorem 6.4.1 (Arrow's Impossibility Theorem [1])**.** *The only social welfare function which satisfies both axioms (i) and (ii) is a dictatorship.*

To explain the connection with the braid arrangement, we note that the complement of the braid arrangement $A_n(\mathbb{R})$ is the disjoint union of several connected components, called the chambers of A_n. Because elements $(v_1, \ldots, v_n)$ of these chambers are in the complement of the braid arrangement, there are no i and j such that $v_i = v_j$ in these elements. Therefore, each of these chambers represents some ordering of the v_i, which corresponds to an overall societal ranking of the n options (option i is preferred to option j if $v_i > v_j$. Let Ch be the set of chambers of the complement of the braid arrangement. Then, a social welfare function of a society of m voters corresponds to a map

$$\Phi : (Ch)^m \to Ch,$$

which satisfies the mathematical interpretation of (i) and (ii). Terao [29] uses the chambers of the complement of A_n to provide a proof for this theorem, by showing that the only social welfare function is a projection, i.e. taking one voter's ranking to be society's ranking.

7 Characteristic Numbers

Another interesting property of hyperplane arrangement intersection types is the number of curves of various degrees they are tangent to. In this setting, an arrangement is tangent to a curve if the curve is an element of the arrangement or if an intersection point of the arrangement lies on the curve. Earlier, we calculated the number of distinct hyperplane arrangements of a fixed combinatorial intersection type that passed through a certain number of points in general position. Now, we are no longer restricted to only placing *points* in general position, we can fix lines and other curves as well. We begin with Theorem 7.0.2 (for a proof, see Fulton [7]).

Theorem 7.0.2. *Let $D_1, \ldots, D_r$ be curves of degree n_i and class m_i in general position. The number of arrangements tangent to $D_1, \ldots, D_r$ is given by expanding the polynomial*

$$\Pi_{i=1}^{r}(m_i\mu + n_i\nu)$$

where $\mu^k\nu^{r-k}$ is to be replaced by the number of arrangements passing through k general points and tangent to $r-k$ general lines.

To make this theorem more applicable, we modify some of the notation to match our own and add a factor to account for arrangements that both pass through points and lie tangent to curves. This yields Theorem 7.0.3, which involves a polynomial we will call the tangency polynomial.

Theorem 7.0.3. *The number of arrangements of ℓ lines with lattice type L with dimension t through p points and tangent to $t-p$ curves of degree d is: $(d\mu + d(d-1)\nu)^{(t-p)}\mu^p$. Here again the monomial $\mu^k\nu^{t-k}$ is to be replaced by the number of arrangements with lattice type L passing through k general points and tangent to $t-k$ lines.*

This gives a formula for the number of arrangements through p points and tangent to $t-p$ curves with respect to L, d, and p. Fulton calls $\mu^k\nu^{t-k}$, which represents the number of arrangements passing though k points and tangent to $t-k$ lines, the "characteristic" of the arrangement. Therefore, to evaluate this polynomial for a particular intersection type, it is necessary to generate the table of characteristic values. Let's look at an example: the pencil of three lines.

Example 7.0.4. Let $\ell = 3$ lines be arranged in a pencil configuration with dimension $t = 5$. Calculate the number of arrangements that pass through p points and $5-p$ curves of degree $d = 1$, with p an integer such that $0 \leq p \leq 5$.

The first characteristic number we calculate is $\mu^5\nu^0$. This is simply the number of pencils of three lines that passes through five points in general position, which we calculated earlier to be $\frac{\binom{5}{2}\binom{3}{2}}{2} = 15$, so $\mu^5\nu^0 = 15$. Next is $\mu^4\nu^1$. Our arrangement must be tangent to this general line, and we defined tangency to mean that two lines in the arrangement must intersect on the general line. In the pencil configuration, all lines meet at a single point, so this lone intersection point must lie on the general line for the arrangement to be tangent to it, as depicted in Figure 19.

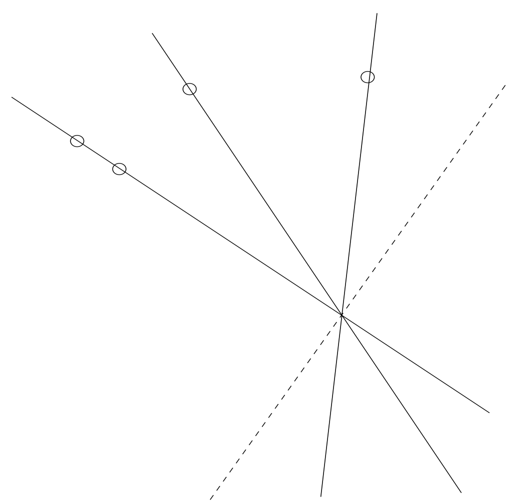

Figure 19: Pencil of three lines passing through four points and tangent to one line (dotted) in general position.

The number of such arrangements is easily calculated. Two of the four general points are chosen for the first line, and the other two lines each pass through one of the general points and the intersection of the first line and the tangent line. The labels on the second and third line are interchangeable, which makes the size of the symmetry group 2. The resulting count is: $\frac{\binom{4}{2}\binom{2}{1}\binom{1}{1}}{2} = 6$.

Solving for the next characteristic number, $\mu^3\nu^2$, is very similar. There are two general lines in this case, and the arrangement must be tangent to both of them. With only one intersection point in the whole arrangement, this means that the intersection point of the pencil must lie at the intersection of the two general lines, shown in Figure 20 on the next page.

Each of the three lines in the arrangement goes through one of the general points and the intersection of the two general lines. No matter what order these three lines are picked in, the arrangement will always be the same. Therefore, after placing the three points and two lines in

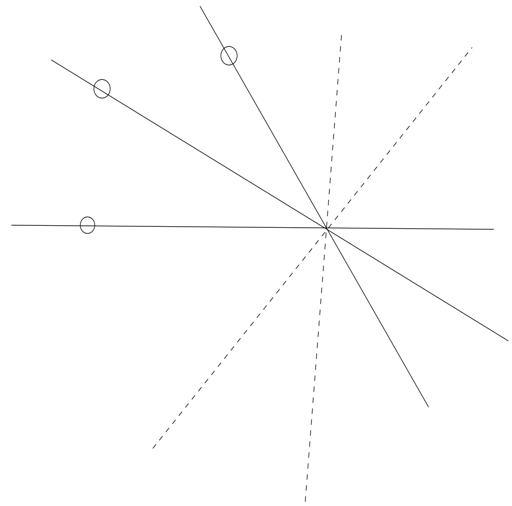

Figure 20: Pencil of three lines passing through three points and tangent to two lines (dotted) in general position.

general position in the projective plane, there is only one pencil that can satisfy our tangency criteria.

The remaining characteristics all have something in common: there are more than two lines that must be tangent to the arrangement. It is quickly apparent that it is impossible to satisfy the tangency criterion. For a pencil of three lines to be tangent to three or more other lines, the other lines must all meet at one point; however, by definition these lines are in general position, so there can not exist such a common intersection point. Therefore, there are no arrangements of pencils of three lines that are tangent to three or more general lines, so these characteristic numbers are 0.

All six characteristic numbers ($\mu^5\nu^0, \mu^4\nu^1, \ldots, \mu^0\nu^5$) are displayed in Table 2.

Points	Lines	Characteristic Number
5	0	15
4	1	6
3	2	1
2	3	0
1	4	0
0	5	0

Table 2: Characteristic numbers for pencils of 3 lines.

Having determined the characteristic numbers for this intersection type, we can begin working with the tangency polynomial to answer enumerative problems by substituting the appropriate characteristic for each $\mu^k\nu^{t-i}$. One such problem is: how many pencils of three lines are tangent to five conics? In this case, $\ell = 3$, $t = 5$, $d = 2$,and $p = 0$, so the number of such arrangements is

$$(2\mu + 2(2-1)\nu)^5\mu^0 = (2\mu + 2\nu)^5 = 32(\mu^5 + 5\mu^4\nu + 10\mu^3\nu^2 + 10\mu^2\nu^3 + 5\mu\nu^4 + \nu^5)$$

$$= 32(15 + 5(6) + 10(1) + 10(0) + 5(0) + 0) = 32(55) = 1760.$$

This answer can be verified independently by combinatorics, as well.

All of the characteristic values for pencils can be calculated for an arbitrary number of lines, making the tangency polynomial easy to use for pencils.

Theorem 7.0.5. *The characteristic numbers for pencils of ℓ lines are shown in Table 3.*

Points	Lines	Characteristic
$\ell + 2$	0	$3\binom{\ell+2}{4}$
$\ell + 1$	1	$\binom{\ell+1}{2}$
ℓ	2	1
$\dots$	$\dots$	$\dots$
1	$\ell + 1$	0
0	$\ell + 2$	0

Table 3: Characteristic numbers for pencils of ℓ lines.

Proof. Let A be an arrangement of ℓ lines forming a pencil.

The characteristic number for $\ell + 2$ points and 0 lines is the same as the count we proved for Theorem 3.1.4:

$$3\binom{\ell+2}{4} = 3(\frac{(\ell+2)!}{4!(\ell-2)!}) = \frac{(\ell+2)!\ell!}{8(\ell!)(\ell-2)!} = \frac{\binom{\ell+2}{2}\binom{\ell}{2}}{2}.$$

The general formula for calculating the characteristic number for $\ell + 1$ points and 1 line in general position is not particularly complicated. Recall from our earlier example that in the case of 4 points and 1 line, we picked two of the points in general position to determine the first line in the arrangement, but after that the remaining lines in the arrangement each had to pass through a point in general position and the intersection of the first line in the arrangement with the general tangent line. Similarly, in the general case, once two of the $\ell + 1$ points are chosen to determine the first of the ℓ lines (contributing a factor of $\binom{\ell+1}{2}$ to the count), the arrangement is fixed and the remaining $\ell - 1$ lines are placed with no choice. This yields a characteristic number of $\binom{\ell+1}{2}$. The reasoning we used in Example 7.0.4 for the case of 3 points and 2 lines is exactly applicable to the general case of ℓ points and 2 lines. The arrangement must be tangent to both general lines and still go through all ℓ points in general position, and there is only one possible pencil of ℓ lines that meets these criteria. Therefore, the characteristic number for ℓ points and 2 lines in general position is always 1. For all of the cases in which more than two lines must be made tangent to the arrangement, the count is 0, because there is only one point of intersection in pencils and that point can not lie on more than two lines in general position. $\square$

Now that we have determined the general formulas for the characteristic numbers of a pencil of ℓ lines, we can use these numbers in the tangency polynomial to solve myriad enumerative problems.

Example 7.0.6. How many arrangements of 5 lines in a pencil pass through 3 points and are tangent to 4 conics placed in general position in the projective plane $\mathbb{P}^2$?

The tangency polynomial, as stated in Theorem 7.0.3, is $(d\mu + d(d-1)\nu)^{(t-p)}\mu^p$, where t is the dimension of the arrangement, p is the number of points in general position, $t-p$ is the number of curves of degree d in general position, and $\mu^k\nu^{t-k}$ is the characteristic number. In this example, $t = 7$, $p = 3$, and $d = 2$, resulting in the following polynomial:

$$\begin{aligned}(2\mu + 2\nu)^4\mu^3 &= 2^4(\mu^4 + 4\mu^3\nu + 6\mu^2\nu^2 + 4\mu\nu^3 + \nu^4)(\mu^3)\\ &= 2^4(\mu^7 + 4\mu^6\nu + 6\mu^5\nu^2 + 4\mu^4\nu^3 + \mu^3\nu^4).\end{aligned}$$

From our formulas in Theorem 3, we calculate the characteristic numbers for a pencil of 5 lines, shown in Table 4.

Points	Lines	Characteristic
7	0	$3\binom{5+2}{4} = 105$
6	1	$\binom{5+1}{2} = 15$
5	2	1
…	…	…
1	6	0
0	7	0

Table 4: Characteristic numbers for pencils of 5 lines

Substituting the appropriate values for $\mu^k\nu^{t-k}$ in the tangency polynomial, we solve for the count:

$$2^4(\mu^7 + 4\mu^6\nu + 6\mu^5\nu^2 + 4\mu^4\nu^3 + \mu^3\nu^4) = 2^4(105 + 4(15) + 6(1) + 4(0) + 1(0)) = 2736.$$

In addition to calculating the characteristic numbers for pencils of ℓ lines, we also investigate the characteristic numbers of lines in general position. This is a much more difficult problem to address, because there are far more points of intersection in the arrangement ($\binom{\ell}{2}$, as opposed to 1), which can be used to satisfy the tangency condition. Using the same counting methods as for the characteristic numbers of pencils, we calculated the characteristic numbers of generic arrangements of 3 lines, displayed in Table 5.

Points	Lines	Characteristic
6	0	15
5	1	30
4	2	48
3	3	57
2	4	48
1	5	30
0	6	15

Table 5: Characteristic numbers for generic arrangements of 3 lines

A very interesting property of the characteristic numbers of generic arrangements of 3 lines is their symmetry, i.e. $\mu^6\nu^0 = \mu^0\nu^6$, $\mu^5\nu^1 = \mu^1\nu^5$, etc. This symmetry comes from the fact that the dual of a generic arrangement of 3 lines through 6 general points is a generic arrangement of 3 lines tangent to 6 general lines, and vice versa. In this case, by the description of dualizing given in Section 2.1, the 6 general points dualize to the 6 general lines, the 3 lines in the arrangement dualize to the 3 double points, and the 3 double points dualize to the 3 lines in the arrangement. The same dual relationship exists for $\mu^5\nu^1/\mu^1\nu^5$ and $\mu^4\nu^2/\mu^2\nu^4$.

While we have a great degree of confidence in the characteristic numbers in Table 5, our results for $\ell = 4$ lines, shown in Table 6 at the top of the next page, are somewhat less certain.

As you can see, there is not nearly the kind of nice symmetrical pattern for the characteristic numbers of generic arrangements as there is for pencils, so we have been unable to this point to find a generalized formula for these characteristic numbers. In fact, we are only able to bound some of the individual characteristic numbers, because these numbers, calculated using the intersection polynomial, contain arrangements with double lines. We are unsure how many such arrangements there are (and in some cases we have excess intersection – higher dimensional extraneous components that count "as a finite number of points" – see Fulton [6]), so we do not know how much we should subtract from the intersection polynomial result. Therefore, we know what the upper bound of the characteristic number is, but not what the actual number is.

There is one very interesting result in the characteristic numbers of 4 generic lines that we can comment on: the number of arrangements tangent to 8 general lines is 16695, the same as the count

Points	Lines	Characteristic
8	0	105
7	1	315
6	2	855
5	3	≤ 2070
4	4	≤ 4010
3	5	≤ 8190
2	6	≤ 13215
1	7	17955
0	8	16695

Table 6: Characteristic numbers for generic arrangements of 4 lines

for the braid arrangement. The reason that these numbers are the same is due to the relationship between the dual of 4 generic lines tangent to 8 general lines and the braid arrangement. When we dualize 4 generic lines that are tangent to 8 general lines, the 8 lines become 8 points in general position, the 4 generic lines become triple points, and the 6 double points in the arrangement become lines. In this dualized arrangement, the 6 dual lines go through the 8 points. Uncovering this relationship presents the possibility that we may be able to find other similar relationships between duals of arrangements, and use this relationship to find the correct characteristic numbers. As tempting as it may seem to assume that the characteristic numbers for generic arrangements of 4 lines exhibit the same kind of symmetry as those for generic arrangements of 3 lines, this is not the case. When working with generic arrangements of 3 lines, it just so happens that the number of double points (which dualize to lines) is equal to the number of lines (which dualize to double points). Because this is the case, generic arrangements of 3 lines through k general points and tangent to $6-k$ general lines always dualize to generic arrangements of 3 lines through $6-k$ general points and tangent to k general lines. However, for generic arrangements of $\ell \geq 4$ lines, there are $\binom{\ell}{2} > \ell$ double points in the arrangement, so generic arrangements of $\ell \geq 4$ lines do not dualize to other generic arrangements of $\ell \geq 4$ lines. Therefore, the characteristic numbers for generic arrangements of 4 or more lines are not symmetrical.

8 Conclusion

In this paper, we answer the following enumerative question: How many line arrangements with a fixed intersection lattice pass through a given number of points in general position in $\mathbb{P}^2$? For some families of arrangements, we have been able to generate explicit formulas to answer this question for an arbitrary number of lines ℓ. We prove that there are

$$\frac{(2\ell)!}{\ell!2^{\ell}} = (2\ell-1)!!$$

generic arrangements of ℓ lines through 2ℓ points in general position, and that there are

$$\frac{1}{2}\binom{\ell+2}{2}\binom{\ell}{2}$$

pencils of ℓ lines through $\ell+2$ points in general position. We also calculate answers to our main question for other interesting intersection lattice types, like the "tie-fighter" of five lines, of which there are 5355 through 8 points in general position. These results can be computed using both combinatorial methods and the intersection polynomial in the Chow ring of $(\mathbb{P}^{2*})^{\ell}$. By taking the coefficient of the appropriate term in this polynomial and dividing by the size of the symmetry group, we arrive at the same answers.

The multivariate Tutte polynomial completely encodes all the information of an intersection lattice or the associated matroid. Many invariants of matroids, like the characteristic polynomial, are evaluations of the multivariate Tutte polynomial. For generic arrangements and pencils, we find connections between evaluations of the Tutte polynomial and the solutions to our enumerative problem.

We also generalize some of our results to enumerative problems of arrangements tangent to arbitrary curves in $\mathbb{P}^2$. To do this, we compute the characteristic numbers for an arrangement and build the tangency polynomial. We compute these characteristic numbers for all pencils and generic arrangements of three lines. Surprisingly, for four generic lines the computations are much more difficult, but we are able to construct upper bounds. The characteristic numbers for generic arrangements of three lines exhibit nice symmetry, which might lead to a generalized notion of duality.

In computing solutions to our main enumerative problem, we investigate the moduli space of various intersection lattices. Each point on one of these moduli spaces corresponds to an arrangement with that intersection lattice. The dimension of a particular intersection lattice's moduli space is equal to the number of points that a finite nonzero number of arrangements of that lattice type pass through. These moduli spaces and their closures can be extremely complicated. However, we can explicitly calculate generators for the ideal of the closure for some complicated examples.

The main results in this paper are related to the braid arrangement A_4. Using both complicated combinatorial and intersection theory methods, we calculate that there are 16695 braid arrangements through 8 points in general position in $\mathbb{P}^2$. In investigating the moduli space of the braid arrangement, we generated a list of minimal defining equations for this moduli space that is different from what is found in the literature. Another interesting result regarding the braid arrangement is that the characteristic number for a generic arrangement of four lines tangent to eight general lines is also 16695. This connection arises because the braid arrangement is the dual of the generic arrangement of four lines tangent to eight general lines.

One of the things that strikes me most in this research is how often intuition can be misleading. Specific enumerative questions are extremely easy to pose, with any number of geometric constraints possible. However, sometimes it takes an extraordinary amount of work just to prove that it is impossible for an arrangement to satisfy these geometric conditions, even when it seems that this shouldn't be the case. Another example of the unpredictable outcome of these problems concerns the braid arrangement and generic arrangement of six lines. A generic arrangement contains no points of higher multiplicity, and every line in the arrangement passes through two points in general position. In the braid arrangement there are four triple points, and multiple complicated constructions. It would seem that there would be more possible generic arrangements than braid arrangements due to the greater degree of "freedom;" however, this is decidedly not the case. There are 10395 generic arrangements of six lines through twelve points and 16695 braid arrangements of six lines through eight points.

9 Open Problems

The results in this paper have spawned many questions for further research. Conceivably, we want a formula to solve the enumerative problem for any intersection lattice type. It is possible that the way in which we will be able to generate these explicit formulas for additional intersection lattice types will not be in combinatorics, as was the case for generic arrangements and pencils, but in the Chow ring.

Similarly, we have found evaluations of the Tutte polynomial that give the correct solution to our main problem for some intersection lattice types, but we do not know how to apply evaluations of the Tutte polynomial to other lattice types. Since the Tutte polynomial encodes the same information as the intersection lattice, we believe there must be a systematic way to solve our enumerative question using the Tutte polynomial. Another area in which we would like to expand our current work is characteristic numbers. Knowing the characteristic numbers for an arrangement is a powerful tool, because it allows us to answer all kinds of enumerative questions involving that arrangement. We have not been able to find formulas for characteristic numbers of families of intersection lattices other than the pencil, but knowing that there is a connection between the counts of an arrangement and its dual (such as the generic arrangement of four lines tangent to eight general lines and the braid arrangement) might give us a new approach to searching for them.

There are many open problems involving the moduli spaces of arrangements with a fixed lattice type. For example, it might be interesting to see if an intersection lattice's moduli space has any singularities, such as self-intersection, and what information that might tell us about arrangements with that intersection lattice. Also, we have found that some moduli spaces, like that of the braid arrangement, are made up of components of different dimensions. Sometimes it is necessary to remove some of these components to eliminate extraneous lattice types, but this is a delicate process. We would like to know more about these components, and the make-up of moduli spaces in general.

A Programming

A.1 Sage and Macaulay2

We have defined two functions, apply and symmetries, to compute and return the number of elements of the symmetry group of ℓ lines that preserve the intersection conditions specified in the intersection polynomial. The input for the apply function is an operation p and a list s, and the output is the list of p applied to all elements of s. For symmetries, the inputs are the list of determinantal conditions lt and a number of lines l, and the outputs are the number of appropriate permutations on the line labels and the list of all of these permutations.

```
def apply(p,s):
    ans = []                      %Creates an empty list into which we can add p
                                  applied to the elements of s
    for i in range(len(s)):       %Indexes s to ensure we account for each of its
                                  elements
        ans.append(p(s[i]))       %For every element of s, adds p applied to that
                                  element to the list ans we created at the beginning
                                  of the function
    return ans                    %At the output we have the list of each element
                                  of s after p has been applied to it
```

```
def symmetries(lt,l):
```

t = apply(set,lt)	%When we input the determinantal conditions lt and then check to see which permutations are one to one and onto from lt onto lt, we want order not to matter. i.e. [1,4,6]=[1,6,4]. To accomplish this in sage, we need to have the elements of lt be a set, but entering sets is a bit tedious, so we enter a list lt and then use the apply function to turn all of the elements of lt into sets
pset = []	%Creates an empty list into which we can add the elements of the symmetry group that preserve the intersections in lt
for p in SymmetricGroup(l):	
flag = 1	%The flag here represents a true or false condition. That is, we start by assigning each p in SymmetricGroup(l) a flag of 1 (or true), and then test it with each element of lt. If there exists an element in the range of p that isn't in lt or there exists an element of lt that isn't in the range of p, then we change the the flag value to 0 (or false). If flag = 0, we break the if statement and move on to the next value of p. If flag remains 1 (i.e. p was one to one and onto), then we add the permutation applied to each of the numbers 1 through l to the list pset
for s in t:	
if set(apply(p,list(s))) not in t:	%We had to convert s back into a list inside the apply function because you can not index a set. Then we turned this into a set to ignore order
flag = 0	
break	
if flag == 1:	
pset.append(apply(p,range(1,l+1)))	
return(len(pset),pset)	%Yields the number of elements of pset (the number of elements in the symmetric group that preserve the intersection points) and those elements of the symmetric group themselves

When working with matroids and the Tutte polynomial, there are a few important calculations that must occur. First, it is important to find the independent sets of an arrangement. To do this, we wrote a function that takes a matrix representation of an arrangement as input, and outputs the list of independent sets. Additionally, to construct the Tutte polynomial we must be able to find the rank of some set of lines. We wrote a rank function that takes as input a list, which represents the grouping of the lines in the arrangement for which we're calculating rank, and the list of all independent sets determined by the independentsets function. The rank function's output is the rank of this list. Finally, we needed a way to build each x_i, so we wrote a variable generating function, which constructs variables of the form x_i for i values from 0 to $\ell - 1$.

```
def independentsets(M):
    c = M.columns()              %Creates a set containing the columns of M
    n = len(c)                   %Determines the number of columns in M
    E = range(n)                 %Creates a list from 0 to n-1
    P = list(powerset(E))        %Generates the list of all subsets of {0,...,n-1}
    I = []                       %Creates an empty list that will eventually be the list of independent sets
    while not(P==[]):            %Checks to make sure P isn't empty
        x = P[0]                 %Takes the first element of P, some collection of numbers between 0 and n-1, which represents some subset of the columns of M
        mat = []
        for y in x:
            mat.append(c[y])     %Each number in x is the label of a column in M. This step uses those labels to retrieve the columns they represent and add them to the list "mat"
        W = matrix(mat)          %Converts "mat" to a matrix, with rows that are columns in M
        Q = transpose(W)         %Transposes W, so that columns in Q are columns in M
        if Q.rank() == len(x):   %Sage's built-in rank function calculates rank of Q
            I.append(x)          %This step uses the fact that if the rank of a matrix is equal to its number of columns, then those columns are independent. Therefore, that subsets of columns in M is independent, so we add the subset "x" to the list of independent sets
            P.remove(x)          %Recall that x was defined as being the first element of P. When we remove x from P at the end of the loop, we make a new element of P the first element and allow the loop to continue
        else:                    %If the rank of Q is not equal to its number of columns, then they are not independent
            EE = range(n)        %Creates a new list of numbers from 0 to n-1
```

```
        for j in x:
            EE.remove(j)                 %Removes the elements of x (some subset of (0,...,n-1))
                                         from EE, because we know that those columns of M repre-
                                         sented by x are dependent
        for i in list(powerset(EE)):     %Generates the powerset of the numbers from 0 to n-1 that
                                         weren't in x
            if list(set(x).union(set(i))) in P:
                                         %Takes the union of x with the elements of the powerset of
                                         EE, and checks to see if those unions are elements of P
                P.remove(list(set(x).union(set(i))))
                                         %This step is a time-saving measure for the program. It
                                         uses the fact that if a subset of columns of M is dependent,
                                         then any subset of columns of M containing that depen-
                                         dent subset must also be dependent. This prevents us from
                                         having to check every single subset of columns of M for
                                         dependence, because as soon as we find a single dependent
                                         subset we are able to eliminate a good number of larger
                                         subsets
    return I                             %Returns independent subsets of columns of M
```

```
def rank(a,S):
    b = set(a)                           %Turns list "a" into a set
    c = 0                                %This value c will eventually be the rank of a
    R = apply(set,S)                     %Turns the list of independent sets S into a set
    for k in R:                          %Takes each of the independent sets one at a time
        if len(k.intersection(b)) ≥ c:   %This step uses the fact that the rank of "a" is equivalent to
                                         the size of its maximum intersection with independent sets
                                         of M
            c = len(k.intersection(b))   %We calculate the size of these intersections for each ele-
                                         ment of S and keep track of the greatest one by setting it
                                         equal to c. Once we've tested each subset, the final value
                                         of c is the rank of "a"
    return c                             %Returns the rank of "a"
```

def varx(i):	
return 'x_'+str(i)	%Oftentimes we work with variables of the form x_i. Sage does not recognize these as variables if inputted directly, so we wrote a program to create objects of this form
def variables(l):	%This program takes the objects we create using varx and makes them variables. This is unusual in that SAGE usually requires the variables to be hard coded into a script – you can't usually have a "variable" number of variables
p = []	%Creates an empty list that will eventually be the list of variables $x_0, x_2, \ldots, x_{l-1}$
for i in range(l):	
p.append(varx(i))	%Adds x_i to the list of variables p for each i from 0 to l-1
v = apply(var,p)	%Uses the apply function to make each of the elements of p become a variable
return v	%Returns a list of variables from x_0 to x_{l-1}

Using the above functions, we made a function that takes a matrix M representing an arrangement and creates that arrangement's Tutte polynomial.

def Tutte(M):	%Input is the matrix representation of a hyperplane arrangement
l = len(M[0])	%Calculates the number of columns in the first row of M
S = independentsets(M)	%Determines the independent subsets of columns of M
P = list(powerset(range(0,l)))	%Creates the list of subsets of (0,...,l-1)
R = apply(set,S)	%Changes the list of independent sets of M into a set
V = variables(l)	%Makes variables from x_0 to x_{l-1}
q = var('q')	%Creates the variable q to be used in the Tutte polynomial
c = 0	%Creates what will eventually be the Tutte polynomial
for i in P:	%Tests each element of the powerset of l
u = set(i)	%Changes a list of labels of columns of M into a set
e = rank(u,R)	%Calculates the rank of that subset of columns of M

a = q^e	%Creates part of the term of the Tutte polynomial corresponding to the particular collection of lines (columns of M) i
for j in i:	%Pulls line labels from i one at a time
a = a*(V[j])	%Builds the term of the Tutte polynomial corresponding to i line by line, pulling variables from the list of variables V
c = c+a	%Once this term "a" is done being constructed, we add it to the existing iteration of the Tutte polynomial "c." Then we construct another term, corresponding to the next element of the powerset
return c	%Returns the final Tutte polynomial

Finally, having determined what the Tutte polynomial is, we want to find soluntions for q and the x_i's that make an arrangement's Tutte polynomial equal to its count. This function's inputs are the Tutte polynomial, the number of lines, the count, and upper and lower bounds for q and the x_i's. In this function, all x_i have the same bound.

def Tuttesolns(T,l,count,qlow,qup,xlow,xup):	%The inputs to this program are the Tutte polynomial T, the number of lines l, and lower and upper bounds for q and x. Setting wide bounds allows for more solutions to be computed, but for faster calculation it is useful to set relatively narrow bounds
solutions=[]	%Creates an empty list to which solutions are added
for z in range(qlow,qup+1):	%Ensures that the Tutte polynomial is evaluated for each value of q between qlow and qup
vars = variables(l)	%Creates a list of variables from 0 to l-1
for u in range((xup-xlow+1)^l):	
S = T(q=z)	%Evaluates the Tutte polynomial for the current value of q and renames it S
t = ZZ(u).digits(base = xup-xlow+1, padto = l, digits = tuple(range(xlow,xup+1)))	%Generates l-tuples (sequences of l numbers) whose elements are integers within the x bounds. These groups of l numbers will be substituted for the l variables x_i to evaluate the Tutte polynomial
m = len(t)	%Calculates the length of t, which is l

while m $>$ 0:	%Checks the value of m to see if it's positive. This while loop ensures that every element of our l-tuple t is substitued into the Tutte polynomial for the appropriate variable
S = S.subs(vars[m-1] == t[m-1])	%Substitutes the (m-1)th element of t for x_{m-1} in the Tutte polynomial
m = m-1	%Having evaluated the Tutte polynomial at x_{m-1}, we bring the value of m down by 1 and run it through the while loop again. If we just evaluated x_0, the while loop will end
if S == count:	%After the while loop has completed, we will have substitued integer values for q and all x_i. The resulting value of the Tutte polynomial will be some integer S. We entered the count, which we calculated using combinatorics or some other method, as an input to the function, and now we simply check and see if these values are the same
solutions.append([z,t])	%If the count is equal to the evaluation of the Tutte polynomial, we record the corresponding q value and x_i values. At this point, we have tested all possible x_i values after setting a single q value. Therefore, the function now picks the next value for q and retests all possible x_i values
return solutions	%After testing all values for q within in the q-bounds and x_i within the x-bounds, we return the list of values for these variables that yielded our desired count

References

[1] Kenneth Arrow. A difficulty in the theory of social welfare. *Journal of Political Economy*, 58:328–346, 1950.

[2] William F. Basener. *Topology and its applications*. Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, 2006.

[3] F. M. Dekking, C. Kraaikamp, H. P. Lopuhaä, and L. E. Meester. *A modern introduction to probability and statistics*. Springer Texts in Statistics. Springer-Verlag London Ltd., London, 2005. Understanding why and how.

[4] David Eisenbud and Joe Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[5] Michael Farber, Mark Grant, and Sergey Yuzvinsky. Topological complexity of collision free motion planning algorithms in the presence of multiple moving obstacles. In *Topology and robotics*, volume 438 of *Contemp. Math.*, pages 75–83. Amer. Math. Soc., Providence, RI, 2007.

[6] William Fulton. *Introduction to intersection theory in algebraic geometry*, volume 54 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1984.

[7] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[8] Robert Ghrist. Configuration spaces, braids, and robotics. In *Braids*, volume 19 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 263–304. World Sci. Publ., Hackensack, NJ, 2010.

[9] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[10] Paul Hacking, Sean Keel, and Jenia Tevelev. Compactification of the moduli space of hyperplane arrangements. *J. Algebraic Geom.*, 15(4):657–680, 2006.

[11] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[12] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[13] David Hilbert. Mathematical problems. *Bull. Amer. Math. Soc. (N.S.)*, 37(4):407–436 (electronic), 2000. Reprinted from Bull. Amer. Math. Soc. **8** (1902), 437–479.

[14] Markus Hohenwarter et al. Geogebra. Available at http://www.geogebra.org/cms/en.

[15] M. M. Kapranov. Chow quotients of Grassmannians. I. In *I. M. Gel′fand Seminar*, volume 16 of *Adv. Soviet Math.*, pages 29–110. Amer. Math. Soc., Providence, RI, 1993.

[16] Sheldon Katz. *Enumerative geometry and string theory*, volume 32 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2006. IAS/Park City Mathematical Subseries.

[17] Steven L. Kleiman. Problem 15: rigorous foundation of Schubert's enumerative calculus. In *Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Northern Illinois Univ., De Kalb, Ill., 1974)*, pages 445–482. Proc. Sympos. Pure Math., Vol. XXVIII. Amer. Math. Soc., Providence, R. I., 1976.

[18] N. E. Mnëv. Varieties of combinatorial types of projective configurations and convex polyhedra. *Dokl. Akad. Nauk SSSR*, 283(6):1312–1314, 1985.

[19] James R. Munkres. *Elementary differential topology*, volume 1961 of *Lectures given at Massachusetts Institute of Technology, Fall*. Princeton University Press, Princeton, N.J., 1963.

[20] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[21] Jürgen Richter-Gebert. *Perspectives on projective geometry*. Springer, Heidelberg, 2011. A guided tour through real and complex geometry.

[22] Richárd Rimányi, László M Fehér, and András Némethi. Equivariant classes of matrix matroid varieties. arXiv:0812.4871v1, December 2008.

[23] Gregory G. Smith and Bernd Sturmfels. Teaching the geometry of schemes. In *Computations in algebraic geometry with Macaulay 2*, volume 8 of *Algorithms Comput. Math.*, pages 55–70. Springer, Berlin, 2002.

[24] Karen E. Smith, Lauri Kahanpää, Pekka Kekäläinen, and William Traves. *An invitation to algebraic geometry*. Universitext. Springer-Verlag, New York, 2000.

[25] Alan D. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In *Surveys in combinatorics 2005*, volume 327 of *London Math. Soc. Lecture Note Ser.*, pages 173–226. Cambridge Univ. Press, Cambridge, 2005.

[26] Richard P. Stanley. An introduction to hyperplane arrangements. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 389–496. Amer. Math. Soc., Providence, RI, 2007.

[27] W. A. Stein et al. *Sage Mathematics Software*. The Sage Development Team, 2012. http://www.sagemath.org.

[28] Hiroaki Terao. Moduli space of combinatorially equivalent arrangements of hyperplanes and logarithmic Gauss-Manin connections. *Topology Appl.*, 118(1-2):255–274, 2002. Arrangements in Boston: a Conference on Hyperplane Arrangements (1999).

[29] Hiroaki Terao. Chambers of arrangements of hyperplanes and Arrow's impossibility theorem. *Adv. Math.*, 214(1):366–378, 2007.

[30] Ravi Vakil. Murphy's law in algebraic geometry: badly-behaved deformation spaces. *Invent. Math.*, 164(3):569–590, 2006.

[31] Sergey Yuzvinsky. Free and locally free arrangements with a given intersection lattice. *Proc. Amer. Math. Soc.*, 118(3):745–752, 1993.